ALGEBRA OF DIMENSION THEORY

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Abstract. The dimension algebra of graded groups is introduced. With the help of known geometric results of extension theory that algebra induces all known results of the cohomological dimension theory. Elements of the algebra are equivalence classes $\dim(A)$ of graded groups A. There are two geometric interpretations of those equivalence classes:
1. For pointed CW complexes K and L, $\dim(H_*(K)) = \dim(H_*(L))$ if and only if the infinite symmetric products $SP(K)$ and $SP(L)$ are of the same extension type (i.e., $SP(K) \in AE(X)$ iff $SP(L) \in AE(X)$ for all compact X).
2. For pointed compact spaces X and Y, $\dim(\mathcal{H}^{-*}(\mathcal{X})) = \dim(\mathcal{H}^{-*}(\mathcal{Y}))$ if and only if X and Y are of the same dimension type (i.e., $\dim_G(X) = \dim_G(Y)$ for all Abelian groups G).

Dranishnikov's version of Hurewicz Theorem in extension theory becomes $\dim(\pi_*(K)) = \dim(H_*(K))$ for all simply connected K.

The concept of cohomological dimension $\dim_A(X)$ of a pointed compact space X with respect to a graded group A is introduced. It turns out $\dim_A(X) \leq 0$ iff $\dim_{A(n)}(X) \leq n$ for all $n \in \mathbf{Z}$. If A and B are two positive graded groups, then $\dim(A) = \dim(B)$ if and only if $\dim_A(X) = \dim_B(X)$ for all compact X.

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1. Introduction

Notation 1.1. *Throughout the paper K, L, and M are reserved for pointed CW complexes. $SP(K)$ (see [1], p.168) is the* **infinite symmetric product** *of a pointed CW complex K. X and Y are general topological spaces (quite often compact or compact metrizable). We will frequently omit coefficients in the case of integral homology and cohomology. Thus, $H_n(K;\mathbf{Z})$ will be shortened to $H_n(K)$ and $H^n(X;\mathbf{Z})$ will be shortened to $H^n(X)$.*

In the paper we consider **graded groups** A indexed by integers. The n-th term of A will be denoted by $A(n)$, so $A = \{A(n)\}_{n\in\mathbf{Z}}$.

Here are the main examples of graded groups from the point of view of this paper.

Definition 1.2. The **homology graded group** $H_*(K)$ of a pointed CW complex K defined by $(H_*(K))(n) = H_n(K)$ for $n \in \mathbf{Z}$.

The **reversed cohomology graded group** $H^{-*}(X)$ of a pointed compact space X is defined by $(H^{-*}(X))(n) = H^{-n}(X)$ for $n \in \mathbf{Z}$.

The **reversed total cohomology graded group** $\mathcal{H}^{-*}(\mathcal{X})$ of a pointed compact space X is defined by declaring $(\mathcal{H}^{-*}(\mathcal{X}))(\backslash)$ to be the direct sum of all $H^{-n}(X, A)$, where A ranges over all pointed closed subsets of X (the total cohomology group was introduced by Shchepin [19] without reversing indices).

The main concept of the paper is the **homological dimension** $\dim_G(A)$ of a graded group A with respect to an Abelian group G. In the case of $A = H_*(K)$ it is equal to the homological dimension $\dim_G(K)$ introduced in [13] as the supremum of n such that $H_k(K;G) = 0$ for all $k < n$. In the case of $A = \mathcal{H}^{-*}(\mathcal{X})$ it is equal to the negative of the cohomological dimension $\dim_G(X)$ of X (it is the infimum of n such that $K(G,n)$ is the absolute extensor of X).

Using the homological dimension of graded groups we give a uniform description of the Bockstein Theory, the Bockstein Algebra (see [16]), and of the Dual Bockstein Algebra (see [9]).

We will use the following results of geometric nature.

Theorem 1.3. *(see [4]) Suppose K is a pointed connected CW complex. If X is finite-dimensional and $K(\pi_i(K), i) \in AE(X)$ for each $i \geq 1$, then $K \in AE(X)$.*

Of major importance to us is the following result of Dranishnikov:

Theorem 1.4. *(see [6]) Suppose K is a pointed CW complex. If X is compact and $K \in AE(X)$, then $SP(K) \in AE(X)$.*

Since $SP(K)$ is homotopy equivalent to the weak product of Eilenberg-MacLane spaces $K(\tilde{H}_i(K), i)$ (see [1], Corollary 6.4.17 on p.223) one has the following:

Theorem 1.5. *(see [6]) Suppose K is a pointed CW complex. If X is compact, then $SP(K) \in AE(X)$ is equivalent to $K(\tilde{H}_i(K), i) \in AE(X)$ for all $i \geq 0$.*

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2. Algebras with valuations

The basic algebraic structure underlying most of algebraic topology is that of a family of homeomorphism classes of pointed spaces belonging to some specialized class closed under operations of wedge $X \vee Y$ and the smash product $X \wedge Y$. Both operations are commutative, associative, have neutral elements, and one has the distributivity $X \wedge (Y \vee Z) = (X \wedge Y) \vee (X \wedge Z)$. Let us call it the **Standard Algebra**. Notice that the wedge plays the role analogous to the direct sum of vector spaces in the sense that $Map(X \vee Y, Z) = Map(X, Z) \times Map(Y, Z)$, and the smash product plays the role of the tensor product of vector spaces in the sense that $Map(X \wedge Y, Z) = Map(X, Map(Y, Z))$. The formal categorical analogy with linear algebra can be worded as follows: the wedge functor is adoint to the cartesian product functor, and the smash product functor is adjoint to the *hom* functor sending X to $Map(X, Z)$.

Suppose we have an algebra $\mathcal{A}$ and a **discrete valuation** $v(X)$ associated with elements of $\mathcal{A}$, so that the following conditions are satisfied:

1. $v(X) \in \mathbf{Z} \cup \{\pm\infty\}$.
2. $v(X \vee Y) = \min(v(X), v(Y))$.
3. $v(X \wedge Y) \geq v(X) + v(Y)$.

Using valuation v one can establish a new equivalence relation between elements of $\mathcal{A}$ and obtain its quotient $\mathcal{A}/\sim$. Here is how it goes: $X \sim Y$ means $v(X \wedge Z) = v(Y \wedge Z)$ for all $Z \in \mathcal{A}$. It is easy to check that $[X] \vee [Y] := [X \vee Y]$ and $[X] \wedge [Y] := [X \wedge Y]$ are well-defined. Obviously, the valuation v factors through $\mathcal{A}/\sim$. Moreover, in the new algebra, $v(a \cdot c) = v(b \cdot c)$ for all c implies $a = b$.

Let us point out similarity of our discrete valuations to those used in classical algebra (see [15]). There, v is defined on $K \setminus \{0\}$, where K is a field and the following conditions are satisfied:

1. $v(a) \in \mathbf{Z}$.
2. $v(a + b) \geq \min(v(a), v(b))$.
3. $v(a \cdot b) = v(a) + v(b)$.

Let us present three basic examples.

Kuzminov Algebra

$\mathcal{A}$ is the subalgebra of the Standard Algebra consisting of all pointed metrizable compacta of finite dimension and $v(X)$ is the negative of the covering dimension of X. We will call the resulting quotient algebra $\mathcal{A}/\sim$ the **Kuzminov Algebra**.

Shchepin Algebra

$\mathcal{A}$ is the subalgebra of the Standard Algebra consisting of all pointed CW complexes and $v(K)$ is the **connectivity index** $cin(K)$ of K, i.e. the supremum of n so that $H_k(K) = 0$ for all $k < n$. We will call the resulting quotient algebra $\mathcal{A}/\sim$ the **Shchepin Algebra** as its roots can be found in [19].

Dimension Algebra of Graded Groups

The elements of $\mathcal{A}$ are isomorphism classes of graded groups A. The wedge $A \vee B$ is simply the direct sum of A and B, and the product $A \wedge B$ is defined so that:

1. $X \to H^{-*}(X)$ is a homomorphism of the Standard Algebra of pointed compact spaces to $\mathcal{A}$.
2. $K \to H_*(K)$ is a homomorphism of the Standard Algebra of pointed CW complexes to $\mathcal{A}$.

The valuation is the connectivity index $cin(A)$ of a graded group A. It is the supremum of n such that $A_i = 0$ for all $i < n$. The quotient algebra is called the **Dimension Algebra of Graded Groups** (denoted by $\mathcal{DGG}$) and will be introduced in Section 6.

3. Extension algebras

In this section we introduce geometrically two important algebras related to extension properties of pointed countable CW complexes.

Recall the following definition from [9].

Definition 3.1. Let $\mathcal{C}$ be a class of compact spaces. Given two pointed CW complexes K and L, $K \sim_{\mathcal{C}} L$ means that the subclass of $\mathcal{C}$ consisting of all X such that $K \in AE(X)$ is identical the subclass of $\mathcal{C}$ consisting of all X such that $L \in AE(X)$. In particular, $K \sim_X L$ means $K \sim_{\{X\}} L$.

The following result follows from Proposition 3.3 of [9]. However, since it is purely geometric we repeat its proof here.

Lemma 3.2. *Suppose K_1, K_2, L_1, L_2 are pointed countable CW complexes. If $K_1 \sim_X K_2$ and $L_1 \sim_X L_2$ for all compacta (respectively, all finite-dimensional compacta) X, then $\Sigma(K_1 \wedge L_1) \sim_X \Sigma(K_2 \wedge L_2)$ for all compacta (respectively, all finite-dimensional compacta) X.*

Proof. It suffices to show that $\Sigma(K_1 \wedge L_1) \in AE(X)$ implies $\Sigma(K_2 \wedge L_2) \in AE(X)$ for all compacta (respectively, finite-dimensional compacta) X. Since $\Sigma(K \wedge L)$ is homotopy equivalent to the join $K * L$ of K and L, X can be expressed as the union of two of its subsets A and B so that $K_1 \in AE(A)$,

$L_1 \in AE(B)$, and A is a countable union of closed subsets of X (see [5]). Therefore $K_2 \in AE(A)$ and, by Olszewski's Completion Theorem (see [18]) there is a G_δ-subset A' of X containing A such that $K_2 \in AE(A')$. Now, $X \setminus A'$ is an F_σ-subset of X satisfying $L_1 \in AE(X \setminus A')$. Therefore $L_2 \in AE(X \setminus A')$ and the main result of [11] says that $K_2 * L_2 \in AE(X)$ which is the same as saying $\Sigma(K_2 \wedge L_2) \in AE(X)$. □

Corollary 3.3. *Suppose K_1, K_2, L_1, L_2 are pointed countable CW complexes.*

1. If $K_1 \sim_X L_1$ for all compacta (respectively, all finite-dimensional compacta) X, then $\Sigma(K_1) \sim_X \Sigma(L_1)$ for all compacta (respectively, all finite-dimensional compacta) X.

2. Suppose $\Sigma^m(K_1) \sim_X \Sigma^m(L_1)$ for some $m \geq 0$ and for all compacta (respectively, all finite-dimensional compacta) X. If $\Sigma^n(K_2) \sim_X \Sigma^n(L_2)$ for some $n \geq 0$ and for all compacta (respectively, all finite-dimensional compacta) X, then $\Sigma^{m+n+1}(K_1 \wedge L_1) \sim_X \Sigma^{m+n+1}(K_2 \wedge L_2)$ for all compacta (respectively, all finite-dimensional compacta) X.

Proof. Put $L = S^0$. Since $K \wedge L = K$ for all K, 1) follows from 3.2.

2) also follows from 3.2 as $\Sigma(\Sigma^m(K) \wedge \Sigma^n(L))$ is homotopy equivalent to $\Sigma^{m+n+1}(K \wedge L)$. □

Corollary 3.3 means that the following definitions make sense.

Definition 3.4. Consider the set of homeomorphic classes of all pointed countable CW complexes. Two pointed CW complexes K and L are said to be equivalent if there is $m \geq$ such that $\Sigma^m(K) \sim_X \Sigma^m(L)$ for all finite-dimensional compacta X. The algebra with addition defined by $[K] + [L] = [K \vee L]$ and multiplication defined by $[K] \cdot [L] = [K \wedge L]$ will be called the **Dranishnikov-Dydak Algebra** as it is equivalent to the one introduced in [9].

Definition 3.5. Consider the set of homeomorphic classes of all pointed countable CW complexes. Two pointed CW complexes K and L are said to be equivalent if there is $m \geq$ such that $\Sigma^m(K) \sim_X \Sigma^m(L)$ for all compacta X. The algebra with addition defined by $[K] + [L] = [K \vee L]$ and multiplication defined by $[K] \cdot [L] = [K \wedge L]$ will be called the **Stable Extension Algebra**.

Remark 3.6. Notice that Dranishnikov-Dydak Algebra and the Stable Extension Algebra are not identical. Indeed, $K(Z, n) \sim_X S^n$ for all $n \geq 1$ and all finite dimensional compacta X (this amounts to the Alexandroff Theorem stating that, in the class of finite-dimensional compacta, both covering dimension and the integral cohomological dimension coincide). However, for all $n \geq 2$ and all $m \geq 0$ there exist compacta X such that $\Sigma^m(K(Z, n)) \sim_X \Sigma^m(S^n)$ fails (see Theorem 9.3 of [13]).

4. Homological dimension of graded groups

Definition 4.1. Two graded groups A and B are **isomorphic** (notation: $A \equiv B$) if $A(n)$ is isomorphic to $B(n)$ for each $n \in \mathbf{Z}$.

Given a family $\{A_t\}_{t\in T}$ of graded groups, its **direct sum** $\bigoplus\limits_{t\in T} A_t$ is defined as the graded group whose n-th term is $\bigoplus\limits_{t\in T} A_t(n)$.

Our first step is to generalize the concept of smash product ($K \wedge L$ for pointed CW complexes and $X \wedge Y$ for pointed compact spaces) to graded groups.

Definition 4.2. Given graded groups A and B their **smash product** $A \wedge B$ is defined by $(A\wedge B)(n) = \bigoplus\{A(k)\otimes B(n-k) \mid k \in \mathbf{Z}\} \oplus \bigoplus\{A(k) * B(n-k-1) \mid k \in \mathbf{Z}\}$.

Definition 4.3. The **suspension operator** Σ^k, $k \in \mathbf{Z} \cup \{\pm\infty\}$, on graded groups is defined as follows:

1. if k is an integer, then $\Sigma^k(A)(n) = A(n-k)$ for all n.
2. if $k = -\infty$, then $\Sigma^k(A)(n) = \bigoplus\limits_{m\in\mathbf{Z}} A(m)$ for all n.
3. if $k = \infty$, then $\Sigma^k(A)(n) = 0$ for all n.

Remark 4.4. Notice that $\Sigma^1(H_*(K)) = H_*(\Sigma(K))$ for pointed CW complexes and $\Sigma^{-1}(H^{-*}(X)) = H^{-*}(\Sigma(X))$ for pointed compact spaces.

Notation 4.5. *When convenient, a group G will be identified with the graded group A so that $A(0) = G$ and $A(k) = 0$ if $k \neq 0$. In particular, we will talk about graded groups $\Sigma^k(G)$ if G is a group. One should think of groups as analogs of Moore spaces.*

Remark 4.6. In view of 4.5 the smash product $A \wedge G$ of A and a group G can be viewed as $(A \otimes G) \oplus \Sigma(A * G)$.

Definition 4.7. Suppose A is a graded Abelian group and G is an Abelian group. The **homological dimension** $\dim_G(A) \in \mathbf{Z} \cup \{\pm\infty\}$ of A is defined as

$$\sup\{k \in \mathbf{Z} \cup \{\pm\infty\} \mid (A \wedge G)(n) = 0 \text{ for all } n < k\}.$$

$\dim(A) \leq \dim(B)$ means that $\dim_F(A) \leq \dim_F(B)$ for any Abelian group F.

Proposition 4.8. *a. For every graded group A there is a free chain complex C such that $A = H_*(C)$.*

b. Suppose A is the homology $H_*(C)$ of a free chain complex C, B is the homology $H_*(D)$ of a free chain complex D, and G is an Abelian group. The following isomorphisms hold:

1. *$A \wedge G \equiv H_*(C \otimes G)$,*
2. *$A \wedge B \equiv H_*(C \otimes D)$.*

Proof. Given a graded group A we can treat it as a trivial chain complex (i.e., all the boundary homomorphisms are trivial). Notice that $A = H_*(A)$ in such case. Lemma 12 in [20] (p.225) says that there is a free approximation C of A. In particular, $H_*(C) = H_*(A) = A$.

1) is a consequence of the Universal-Coefficient Theorem for homology (see [20], Theorem 8 on p.222). 2) is a consequence of the Künneth Formula for homology (see [20], Theorem 3 on p.230). $\square$

Proposition 4.9. *Smash product is associative and commutative.*

Proof. The commutativity of smash product follows from commutativity of the tensor product and the torsion product. Given three graded groups A_i, $1 \le i \le 3$, choose free chain complexes C_i such that $A_i = H_*(C_i)$ for $1 \le i \le 3$ (see a) of 4.8). Using 4.8 notice that each of $(A_1 \wedge A_2) \wedge A_3$ and $A_1 \wedge (A_2 \wedge A_3)$ equals $H_*(C_1 \otimes C_2 \otimes C_3)$ (use associativity of the tensor product). $\square$

Corollary 4.10. *Suppose A_1, A_2 and B_1, B_2 are graded groups. If dim$(A_i) \le$ dim(B_i) for $i \le 2$, then dim$(A_1 \oplus B_1) \le$ dim$(A_2 \oplus B_2)$ and dim$(A_1 \wedge B_1) \le$ dim$(A_2 \wedge B_2)$.*

Proof. The inequality $\dim(A_1 \oplus B_1) \le \dim(A_2 \oplus B_2)$ follows from $(A \oplus B) \wedge G \equiv (A \wedge G) \oplus (B \wedge G)$ for any graded groups A, B and any group G.

First, consider the case $B_1 = B_2 = B$ is a group. Notice that $A_1 \le A_2$ means $\dim_{\mathbf{Z}}(A_1 \wedge G) \le \dim_{\mathbf{Z}}(A_2 \wedge G)$ for all groups G. Therefore $\dim_{\mathbf{Z}}(A_1 \wedge B) \le \dim_{\mathbf{Z}}(A_2 \wedge B)$ for all $B = \Sigma^n(G)$, where $n \in \mathbf{Z}$ and G is a group. Since every graded group C is a direct sum of $\Sigma^n(C(n))$, $n \in \mathbf{Z}$, one gets $\dim_{\mathbf{Z}}(A_1 \wedge C) \le \dim_{\mathbf{Z}}(A_2 \wedge C)$ for all C. In particular, for $C = B \wedge G$ this amounts to $\dim_G(A_1 \wedge B) \le \dim_G(A_2 \wedge B)$.

The general case follows from commutativity of the smash product and the special case; $\dim(A_1 \wedge B_1) \le \dim(A_1 \wedge B_2) \le \dim(A_2 \wedge B_2)$. $\square$

For any two Abelian groups F and G one can discuss the concept of $\dim(G) \le \dim(F)$ and of $\dim(G) = \dim(F)$ using Convention 4.5.

Proposition 4.11. *For any two Abelian groups F and G, the dimension dim$_G(F)$ can attain only three values; 0, 1, and ∞.*

1. *dim$_G(F) = \infty$ if and only if $G \otimes F = 0 = G * F$.*
2. *dim$_G(F) = 1$ if and only if $G \otimes F = 0$ and $G * F \ne 0$.*
3. *dim$_G(F) = 0$ if and only if $G \otimes F \ne 0$.*

Proof. Let $A = F$ and $B = A \wedge G$. Notice that $B(1) = F * G$, $B(0) = F \otimes G$, and $B(n) = 0$ for $n \ne 0, 1$. $\square$

Corollary 4.12. *The following conditions are equivalent for any two Abelian groups F and G:*

1. *dim$(F) \le$ dim(G).*
2. *dim$_F(A) \le$ dim$_G(A)$ for any graded group A.*
3. *If $H \otimes F = 0$, then $H \otimes G = 0$ for every Abelian group H. If $H \otimes F = 0$ and $H * F = 0$, then $H \otimes G = 0$ and $H * G = 0$ for every Abelian group H.*

Proof. 1) $\iff$ 2) is immediate from 4.11. 3) $\implies$ 2) is obvious and 2) $\implies$ 3) by using $A = H$. □

Remark 4.13. Notice how these concepts relate to Shchepin's [19] factorial domination.

5. Extension theory and homological dimension

Corollary 5.1. *Let G be an Abelian group. If K and L are pointed CW complexes, then*
1. *$H_*(K;G) \equiv H_*(K) \wedge G$.*
2. *$H_*(K \wedge L) \equiv H_*(K) \wedge H_*(L)$.*

Proof. For any CW complex M one has the free chain complex $C(M)$ so that $C(M)(n) = C_n(M)$ is the group of cellular n-chains. $H_*(M)$ is simply $H_*(C(M))$ and $H_*(M;G)$ is defined as $H_*(C(M) \otimes G)$. Thus, 1) of 5.1 follows from 4.8. 2) of 5.1 follows from the Künneth formula for singular homology (see Theorem 10 on p.235 in [20]). □

In [13] the inequality $K \leq_G L$ was defined to mean $\dim_G(K) \leq \dim_G(L)$, where $\dim_G(M)$ is the minimum of n such that $H_k(M;G) = 0$ for all $k < n$.

Corollary 5.2. *Suppose K and L are CW complexes and G is an Abelian group. $K \leq_G L$ if and only if $dim_G(\tilde{H}_*(K)) \leq dim_G(\tilde{H}_*(L))$.*

Given a cohomology theory h^*, by $h^{-*}(X)$ we will denote **the reversed cohomology graded group** defined by $(h^{-*}(X))(n) = h^{-n}(X)$.

Obviously, every homology theory h_* (respectively, cohomology theory h^*) gives rise to a graded group $h_*(X)$ (respectively, $h^{-*}(X)$) for every space X. The following definition generalizes Shchepin's concept of total cohomology (see [19]).

Definition 5.3. Let G be an Abelian group. For any pointed compact space X its **graded total cohomology group** $\mathcal{H}^{-*}(\mathcal{X};\mathcal{G})$ is the direct sum of graded groups $H^{-*}(X,A;G)$ with A ranging over all pointed closed subsets of X.

Theorem 5.4. *Suppose X, Y are pointed compact spaces and G is an Abelian group.*
1. *$\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{G} \equiv \mathcal{H}^{-*}(\mathcal{X};\mathcal{G})$.*
2. *$dim(\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}^{-*}(\mathcal{Y})) = dim(\mathcal{H}^{-*}(\mathcal{X} \wedge \mathcal{Y}))$.*

Proof. 1) follows from the Universal coefficient formula for cohomology.

Recall that 4.11 in [13] implies

$$H^{-n}(X \wedge Y) \equiv \bigoplus_i H^{-i}(X; H^{-(n-i)}(Y)).$$

Using that one can view $\mathcal{H}^{-*}(\mathcal{X}))\wedge\mathcal{H}^{-*}(\mathcal{Y})$ as a direct summand of $\mathcal{H}^{-*}(\mathcal{X}\wedge \mathcal{Y})$ as follows: $\mathcal{H}^{-*}(\mathcal{X}) = \bigoplus_{\mathcal{A}\subset\mathcal{X}} \mathcal{H}^{-*}(\mathcal{X}/\mathcal{A})$, $\mathcal{H}^{-*}(\mathcal{Y}) = \bigoplus_{\mathcal{B}\subset\mathcal{Y}} \mathcal{H}^{-*}(\mathcal{Y}/\mathcal{B})$, and

$$(\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}^{-*}(\mathcal{Y}))(\backslash) \equiv$$

$$\bigoplus_{A\subset X, B\subset Y} \bigoplus_i H^{-i}(X/A; H^{-(n-i)}(Y/B)) \equiv$$
$$\bigoplus_{A\subset X, B\subset Y} H^{-n}(X\wedge Y/(X\wedge B\cup A\wedge Y)).$$

Thus, $\dim(\mathcal{H}^{-*}(\mathcal{X})\wedge\mathcal{H}^{-*}(\mathcal{Y})) \geq \dim(\mathcal{H}^{-*}(\mathcal{X}\wedge\mathcal{Y}))$ holds.

Suppose $\dim_G(\mathcal{H}^{-*}(\mathcal{X})\wedge\mathcal{H}^{-*}(\mathcal{Y})) \geq \backslash$ for some Abelian group G. In view of the above we get that $H^i(X\times Y/(X\times B\cup A\times Y); G) = 0$ for all $i \geq -n$, all closed subsets A of X, and all closed subsets B of Y. Let $Z = X\times Y$. Consider

$$\mathcal{V} = \{V \subset Z \mid V \text{ is open and } H^i(Z, Z-V; G) = 0 \text{ for all } i \geq -n\}.$$

Our goal is to show that $\mathcal{V}$ contains all open sets in Z which would complete the proof of 2). Notice that $U\times V \in \mathcal{V}$ if U is open in X and V is open in Y. Indeed, $Z/(Z-U\times V) = X\times Y/((X-U)\times Y \cup X\times(Y-V))$. The family $\{U\times V \mid U \text{ is open in } X, V \text{ is open in } Y\}$ is denoted by $\mathcal{U}$ and it is a subset of $\mathcal{V}$.

Notice that if $V_1 \subset V_2 \subset \dots$ is an increasing sequence of elements in $\mathcal{V}$, then $\bigcup_j V_j \in \mathcal{V}$. It is so as $H^i(Z, Z-\bigcup_j V_j; G) = \text{dirlim} H^i(Z, Z-V_j; G) = 0$ for all $i \geq -n$. Let us show that $V, W, V\cap W \in \mathcal{V}$ implies $V\cup W \in \mathcal{V}$. Let $A = Z\cup C(Z-V)$ and $B = Z\cup C(Z-W)$ be subsets of the cone $C(Z)$ over Z. Then, $H^i(A; G) = H^i(Z, Z-V; G) = 0$, $H^i(B; G) = H^i(Z, Z-W; G) = 0$ and $H^i(A\cup B; G) = H^i(Z, Z-V\cap W; G) = 0$ for all $i \geq -n$. From the Mayer-Vietoris exact sequence $H^i(A\cup B; G) \to H^i(A; G)\oplus H^i(B; G) \to H^i(A\cap B; G) \to H^{i+1}(A\cup B; G) \to \dots$ we get $H^i(A\cap B; G) = H^i(Z, Z-(V\cup W); G) = 0$ for all $i \geq -n$. Thus, $V\cup W \in \mathcal{V}$. Now, it is easy to show by induction on $m \geq 1$ that the union of m elements from $\mathcal{U}$ belongs to $\mathcal{V}$. Since every open set in Z can be expressed as a union of an increasing sequence of open sets, each of which is a finite union of elements of $\mathcal{U}$, the proof of 2) is complete. □

Recall that, for an unpointed compact space X, its **cohomological dimension** $\dim_G(X)$ can be defined in two equivalent ways (see [16]):

1. As the smallest integer n such that $H^k(X, A; G) = 0$ for all $k \geq n+1$ and all closed subsets A of X.
2. As the smallest integer n such that $K(G, n) \in AE(X)$.

We will define $\dim_G(X)$ for pointed compact spaces using the concept of homological dimension.

Definition 5.5. Suppose X is a pointed compact space. Given an Abelian group G define **the cohomological dimension** $\dim_G(X)$ (also denoted by $d_X(G)$) as $-\dim_G(\mathcal{H}^{-*}(\mathcal{X})) = -\dim_{\mathbf{Z}}(\mathcal{H}^{-*}(\mathcal{X}; \mathcal{G}))$.

Notice that $\dim_0(X) = -\infty$ for any pointed compact space X. Also, $\dim_G(X) = -\infty$ for any G if X is a pointed point. However, in the remaining cases, our concept of the cohomological dimension coincides with the one for the corresponding unpointed space.

Proposition 5.6. *Let G be a non-trivial Abelian group. If X is a compact space containing at least two points, then $dim_G(X, x_0) = dim_G(X)$ for all $x_0 \in X$.*

Proof. Notice that, for $n \geq 1$, the groups $H^n(X, A; G)$ and $H^n(X, A \cup \{x_0\}; G)$ are isomorphic. Consequently, $\dim_G(X) \leq n-1$ is equivalent to $\dim_G(X, x_0) \leq n-1$. Hence, the only case where $\dim_G(X, x_0) \neq \dim_G(X)$ may occur is $\dim_G(X) = 0$. In that case X is totally disconnected, so $H^0(X, x_0; G) \neq 0$ as X contains at least two points. Thus, $\dim_G(X, x_0) = 0$ in that case as well. □

We generalize the concept of cohomological dimension with respect to a group to that of cohomological dimension with respect to a graded group as it simplifies calculations.

Definition 5.7. Suppose X is a pointed compact space. Given a graded group A define **the cohomological dimension** $\dim_A(X)$ as $-\dim_{\mathbf{Z}}(\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{A})$.

Proposition 5.8. *Suppose X is a pointed compact space and A is a graded group. The following conditions are equivalent:*

1. *$dim_A(X) \leq k$.*
2. *$dim_{A(n)}(X) \leq n + k$ for each $n \in \mathbf{Z}$.*

Proof. Let $B = \mathcal{H}^{-*}(\mathcal{X})$. Since $A = \bigoplus\limits_{n\in\mathbf{Z}} \Sigma^n(A(n))$,

$$\dim_A(X) = -\dim_{\mathbf{Z}}(\bigoplus_{n\in\mathbf{Z}} \Sigma^n(A(n) \wedge B) =$$

$$-\inf_{n\in\mathbf{Z}}\{\dim_{\mathbf{Z}}(\Sigma^n(A(n) \wedge B))\} = -\inf_{n\in\mathbf{Z}}\{n - \dim_{A(n)}(X)\}.$$

Thus, $\dim_A(X) \leq k$ iff $n - \dim_{A(n)}(X) \geq -k$ for each n which is equivalent to 2). □

Corollary 5.9. *Suppose X is a pointed compact space and K is a pointed CW complex. The following conditions are equivalent:*

1. *$SP(K) \in AE(X)$.*
2. *$\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}_*(\mathcal{K})$ is non-negative.*

Proof. By 1.5 $SP(K) \in AE(X)$ is equivalent to $\dim_{H_n(K)}(X) \leq n$ for all $n \in \mathbf{Z}$. By 5.8 that is equivalent to $\dim_{H_*(K)}(X) \leq 0$ which is another way of saying that $\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}_*(\mathcal{K})$ is non-negative in view of definition 5.7. □

Corollary 5.10. *Suppose X, Y are pointed compact spaces and K, L are pointed CW complexes. If $SP(K) \in AE(X)$ and $SP(L) \in AE(Y)$, then $SP(K \wedge L) \in AE(X \wedge Y)$.*

Proof. Suppose Z is a compact space and M is a CW complex. 5.9 says that $SP(M) \in AE(Z)$ if and only if $\mathcal{H}^{-*}(\mathcal{Z}) \wedge \mathcal{H}_*(\mathcal{M})$ is non-negative. Thus, both $\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}_*(\mathcal{K})$ and $\mathcal{H}^{-*}(\mathcal{Y}) \wedge \mathcal{H}_*(\mathcal{L})$ are non-negative. Obviously,

their smash product is non-negative which means (see 5.4) that $\mathcal{H}^{-*}(\mathcal{X} \wedge \mathcal{Y}) \wedge \mathcal{H}_*(\mathcal{K} \wedge \mathcal{L})$ is non-negative, i.e., $SP(K \wedge L) \in AE(X \wedge Y)$ by 5.9. $\square$

Remark 5.11. Compare the simplicity of the proof of 5.10 with that of Theorem 5.4 of [10] ($K \in AE(X)$ and $L \in AE(Y)$ imply $SP(K \wedge L) \in AE(X \wedge Y)$ provided X is metrizable, Y is metrizable and σ-compact, K and L are CW complexes).

6. Bockstein groups and Bockstein basis

Definition 6.1. Let $\mathcal{DGG}$ be the **dimension algebra of graded groups**. It consists of equivalence classes of graded groups: $A \sim B$ means $\dim(A) = \dim(B)$. The direct product serves as the sum and the smash product serves as the product in $\mathcal{DGG}$.

Notice that $\mathbf{Z}$ serves as the unit of $\mathcal{DGG}$ ($A \wedge \mathbf{Z} = A$ for all graded groups A). In this section we will:
1. show that all objects of $\mathcal{DGG}$ form a set,
2. find its basis.

Definition 6.2. Let $\mathcal{B}_{\mathcal{G}}$ (the **Bockstein groups**) be the set of groups consisting of the rationals $\mathbf{Q}$, the cyclic groups $\mathbf{Z}/\mathbf{p}$ for all primes $\mathbf{p} > 0$, the quasi-cyclic groups $\mathbf{Z}/\mathbf{p}^\infty$ for all primes $\mathbf{p} > 0$, and $\mathbf{Z}_{(\mathbf{p})}$ ($\mathbf{Z}$ localized at $\mathbf{p}$) for all primes $\mathbf{p} > 0$.

Proposition 6.3. *Let G be an Abelian group. If $0 \to A \to B \to C \to 0$ is short exact sequence of graded groups, then one has a long exact sequence of graded groups*

$$\to \Sigma^{-1}(C)\wedge G \to A\wedge G \to B\wedge G \to C\wedge G \to \Sigma(A)\wedge G \to \Sigma(B)\wedge G \to \Sigma(C)\wedge G \to$$

If C is torsion-free, then one has a short exact sequence

$$0 \to A \wedge G \to B \wedge G \to C \wedge G \to 0.$$

Proof. By Corollary 9 on p.224 in [20] one has an exact sequence $0 \to A*G \to B*G \to C*G \to A \otimes G \to B \otimes B \to C \otimes G \to 0$. If C is torsion-free, then it implies exactness of $0 \to A \wedge G \to B \wedge G \to C \wedge G \to 0$. For arbitrary C one splices $0 \to A*G \to B*G \to C*G \to A \otimes G \to B \otimes B \to C \otimes G \to 0$ to obtain $\to \Sigma^{-1}(C) \wedge G \to A \wedge G \to B \wedge G \to C \wedge G \to \Sigma(A) \wedge G \to \Sigma(B) \wedge G \to \Sigma(C) \wedge G \to$. $\square$

Corollary 6.4. *If $0 \to A \to B \to C \to 0$ is an exact sequence of graded groups then*
1. *$A \oplus C \le B$, $B \oplus \Sigma^{-1}(C) \le A$, and $\Sigma(A) \oplus B \le C$.*
2. *If C is torsion-free, then $\dim(B) = \dim(A \oplus C)$.*
3. *If $\dim(A) = \dim(C)$ and $\dim_{\mathbf{Z}}(A) > -\infty$, then $\dim(B) = \dim(A)$.*

Proof. 1) and 2) follow from 6.3.

Suppose $\dim(A) = \dim(C)$. Therefore $A \le B$ by 1) and if $A < B$, then there is k and an Abelian group G such that $\dim_G(B) \ge k+1$ but $\dim_G(A) =$

k. That however means $(\Sigma(A) \wedge G)(k) = 0$ implying $(C \wedge G)(k) = 0$ and contradicting $\dim_G(C) = \dim_G(A) = k$. □

Given any graded group A, $\mathrm{Tor}(A)$ is defined via $\mathrm{Tor}(A)(n) = \mathrm{Tor}(A(n))$ for each n. Also, $A/\mathrm{Tor}(A)$ is defined via $A/\mathrm{Tor}(A)(n) = A(n)/\mathrm{Tor}(A(n))$ for each n. Since $0 \to \mathrm{Tor}(A) \to A \to A/\mathrm{Tor}(A) \to 0$ is exact and $A/\mathrm{Tor}(A)$ is torsion-free, 6.4 implies the following.

Corollary 6.5. *$dim(A) = dim((A/Tor(A)) \oplus Tor(A))$ for any graded group A.*

Corollary 6.6. *Let $\mathbf{p}$ be a prime.*

1. *$dim(\mathbf{Z}/\mathbf{p}^k) = dim(\mathbf{Z}/\mathbf{p})$ for all $k \geq 1$.*
2. *$\mathbf{Z}/\mathbf{p} \leq H \leq \Sigma(\mathbf{Z}/\mathbf{p})$ for all $\mathbf{p}$-groups H.*
3. *$H \leq \mathbf{Z}/\mathbf{p}$ if and only if H is not divisible by $\mathbf{p}$.*
4. *$dim(H) = dim(\mathbf{Z}/\mathbf{p})$ if H is a $\mathbf{p}$-group not divisible by $\mathbf{p}$.*
5. *$dim(H) = dim(\mathbf{Z}/\mathbf{p}^\infty)$ if $H \neq 0$ is a $\mathbf{p}$-group divisible by $\mathbf{p}$.*
6. *$Z_{(\mathbf{p})} < Z/\mathbf{p} < \mathbf{Z}/\mathbf{p}^\infty < \Sigma(\mathbf{Z}/\mathbf{p})$.*

Proof. Notice that, for every $k \geq 1$, there is an exact sequence $0 \to \mathbf{Z}/\mathbf{p}^k \to \mathbf{Z}/\mathbf{p}^{k+1} \to \mathbf{Z}/\mathbf{p} \to 1$. Therefore, 1) follows from Part 3 of 6.4.

By 1) all finite $\mathbf{p}$-groups $G \neq 0$ satisfy $\dim(G) = \dim(\mathbf{Z}/\mathbf{p})$. Since any $\mathbf{p}$-group is the direct limit of its finite subgroups, 2) follows.

3. If $\mathbf{p} \cdot H \neq H$, then $H \otimes (\mathbf{Z}/\mathbf{p}) = H/\mathbf{p} \cdot H$ is a direct sum of copies of $\mathbf{Z}/\mathbf{p}$. Therefore $H \leq H \wedge (\mathbf{Z}/\mathbf{p}) \leq H \otimes (\mathbf{Z}/\mathbf{p}) \leq \mathbf{Z}/\mathbf{p}$. If $H \leq \mathbf{Z}/\mathbf{p}$, then $H \wedge \mathbf{Z}/\mathbf{p} \leq \mathbf{Z}/\mathbf{p} \wedge \mathbf{Z}/\mathbf{p} \leq \mathbf{Z}/\mathbf{p}$ and $H/\mathbf{p} \cdot H = H \otimes \mathbf{Z}/\mathbf{p} = (H \wedge \mathbf{Z}/\mathbf{p})(0)$ cannot be 0.

4) follows from 2) and 3).

5. If $H \neq 0$ is a $\mathbf{p}$-divisible $\mathbf{p}$-group, it is a direct sum of copies of $\mathbf{Z}/\mathbf{p}^\infty$.

6. $Z_{(\mathbf{p})} \leq Z/\mathbf{p} \leq \mathbf{Z}/\mathbf{p}^\infty \leq \Sigma(\mathbf{Z}/\mathbf{p})$ follows from 2) and 3). $\dim(Z_{(\mathbf{p})}) \neq \dim(Z/\mathbf{p})$ as $Z_{(\mathbf{p})} \otimes \mathbf{Z}/\mathbf{p}^\infty \neq 0$ and $\mathbf{Z}/\mathbf{p} \otimes \mathbf{Z}/\mathbf{p}^\infty = 0$. $\dim(\mathbf{Z}/\mathbf{p}^\infty) \neq \dim(Z/\mathbf{p})$ as $Z/\mathbf{p} \otimes \mathbf{Z}/\mathbf{p} \neq 0$ and $\mathbf{Z}/\mathbf{p} \otimes \mathbf{Z}/\mathbf{p}^\infty = 0$. $\dim(\mathbf{Z}/\mathbf{p}^\infty) \neq \dim(\Sigma(\mathbf{Z}/\mathbf{p}))$ as $\dim_{\mathbf{Z}}(\mathbf{Z}/\mathbf{p}^\infty) = 0 \neq \dim_{\mathbf{Z}}(\Sigma(\mathbf{Z}/\mathbf{p})) = 1$. □

Corollary 6.7. *If G is a torsion-free group, then $dim(G) = dim(\mathbf{Q} \oplus \bigoplus\limits_{\mathbf{p} \in l} \mathbf{Z}_{(\mathbf{p})})$, where $l = \{\mathbf{p} \mid \mathbf{p} \cdot G \neq G\}$.*

Proof. It suffices to show that, for any torsion-free group F, the condition $F \otimes H = 0$ is equivalent to $H = \mathrm{Tor}(H)$ and $\mathbf{p} - \mathrm{Tor}(H) = 0$ for any $\mathbf{p}$ not dividing F. Clearly, if $\mathbf{q} \cdot F = F$, then $F \otimes H = 0$ for any $\mathbf{q}$-group H. If $\mathbf{p} \cdot F \neq F$ and $H \neq 0$ is a $\mathbf{p}$-group, then, in view of $F \leq \mathbf{Z}/\mathbf{p}$ (see 6.6) and $H \leq \Sigma(\mathbf{Z}/\mathbf{p})$ (see 6.6), we get $F \otimes H = F \wedge H \leq (\mathbf{Z}/\mathbf{p}) \wedge \Sigma(\mathbf{Z}/\mathbf{p}) \leq \Sigma^2(\mathbf{Z}/\mathbf{p}) \neq 0$ which implies that $F \otimes H \neq 0$. □

Definition 6.8. Given an Abelian group G define its **Bockstein basis** $\sigma(G)$ as the set of all Bockstein groups H such that $\dim(G) \leq \dim(H)$.

Corollary 6.9. *$dim(G) = dim(\bigoplus\limits_{H \in \sigma(G)} H)$ for any Abelian group G.*

Proof. Since $\dim(G) = \dim((G/\mathrm{Tor}(G)) \oplus \mathrm{Tor}(G))$, 6.6 and 6.7 imply that there is a subset S of $\sigma(G)$ such that $\dim(G) = \dim(\bigoplus\limits_{H\in S} H)$. However, $G \le H$ implies $\dim(G) = \dim(G \oplus H)$, so 6.9 follows. $\square$

Proposition 6.10. *The following conditions are equivalent for any graded Abelian groups A and B:*

1. *$dim(A) \le dim(B)$.*
2. *$dim_G(A) \le dim_G(B)$ for all $G \in \mathcal{B}_{\mathcal{G}}$.*

Proof. Clearly, 1) $\implies$ 2). If $\dim_G(A) \le \dim_G(B)$ for all $G \in \mathcal{B}_{\mathcal{G}}$, then $\dim_G(A) \le \dim_G(B)$ for all G which are direct sums of Bockstein groups. By 6.9, 2) $\implies$ 1). $\square$

The next result shows that our definition of the Bockstein basis coincides with the one in [13] and the only difference with the definitions in [16] or [8] is that the case of $\mathbf{Z}/\mathbf{p}^\infty$ is treated differently.

Proposition 6.11. *Let G be an Abelian group.*

1. *$\mathbf{Q} \in \sigma(G)$ if and only if $G \ne Tor(G)$.*
2. *$\mathbf{Z}_{(\mathbf{p})} \in \sigma(G)$ if and only if $G/\,Tor(G)$ is not divisible by $\mathbf{p}$.*
3. *$\mathbf{Z}/\mathbf{p} \in \sigma(G)$ if and only if G is not divisible by $\mathbf{p}$.*
4. *$\mathbf{Z}/\mathbf{p}^\infty \in \sigma(G)$ if and only if either $\mathbf{Z}_{(\mathbf{p})} \in \sigma(G)$ or $\mathbf{p} - Tor(G) \ne 0$.*

Proof. 1. If $\dim(G) \le \dim(\mathbf{Q})$, then $G \otimes \mathbf{Q} = G \wedge \mathbf{Q} \le \mathbf{Q} \otimes \mathbf{Q} = \mathbf{Q}$, so $G \otimes \mathbf{Q} \ne 0$ and G cannot be a torsion group. If $G \ne \mathrm{Tor}(G)$, then (see 6.5) $\dim(G) = \dim(\mathrm{Tor}(G) \oplus G/\mathrm{Tor}(G)) \le \dim(G/\mathrm{Tor}(G)) \le \dim(\mathbf{Q})$ by 6.7.

2. Suppose $\dim(G) \le \dim(\mathbf{Z}_{(\mathbf{p})})$ and $G/\mathrm{Tor}(G)$ is divisible by $\mathbf{p}$. Therefore $0 = G \otimes \mathbf{Z}/\mathbf{p}^\infty$ and since $G \wedge \mathbf{Z}/\mathbf{p}^\infty \le \mathbf{Z}_{(\mathbf{p})} \wedge \mathbf{Z}/\mathbf{p}^\infty = \mathbf{Z}/\mathbf{p}^\infty$, we arrive at a contradiction. If $G/\mathrm{Tor}(G)$ is not divisible by $\mathbf{p}$, then (see 6.7) $G/\mathrm{Tor}(G) \le \mathbf{Z}_{(\mathbf{p})}$. Since $G \le G/\mathrm{Tor}(G)$, $\mathbf{Z}_{(\mathbf{p})} \in \sigma(G)$.

3. Follows from Part 3 of 6.6.

4. Since $Z_{(\mathbf{p})} < \mathbf{Z}/\mathbf{p}^\infty$, $Z_{(\mathbf{p})} \in \sigma(G)$ implies $\mathbf{Z}/\mathbf{p}^\infty \in \sigma(G)$. If $\mathbf{p} - \mathrm{Tor}(G) \ne 0$, then $G \le \mathrm{Tor}(G) \le \mathbf{p} - \mathrm{Tor}(G) \le \mathbf{Z}/\mathbf{p}^\infty$ by 6.6. Suppose $\dim(G) \le \dim(\mathbf{Z}/\mathbf{p}^\infty)$, $G/\mathrm{Tor}(G)$ is divisible and $\mathbf{p} - \mathrm{Tor}(G) = 0$. Hence $\mathbf{p} \cdot G = G$ and $G * \mathbf{Z}/\mathbf{p} = 0$. Consequently $0 = G \wedge \mathbf{Z}/\mathbf{p} \le \mathbf{Z}/\mathbf{p}^\infty \wedge \mathbf{Z}/\mathbf{p} \ne 0$, a contradiction. $\square$

Definition 6.12. Given a function $\alpha : \mathcal{B}_{\mathcal{G}} \to \mathbf{Z} \cup \{\pm\infty\}$ define $GG(\alpha)$ as $\bigoplus\limits_{H\in\mathcal{B}_{\mathcal{G}}} \Sigma^{\alpha(H)}(H)$.

Proposition 6.13. *For every Abelian group A there is a function $\alpha : \mathcal{B}_{\mathcal{G}} \to \mathbf{Z} \cup \{\pm\infty\}$ such that $dim(A) = dim(GG(\alpha))$.*

Proof. Given a graded group B we define a new graded group B' via $B'(n) = \bigoplus\limits_{i\le n} B(i)$ and notice that $\dim(B) = \dim(B')$.

Given a Bockstein group H there are three possibilities:

a. $H \in \sigma(A'(n))$ for each n. Put $\alpha(H) = -\infty$.

b. $H \notin \sigma(A'(n))$ for each n. Put $\alpha(H) = \infty$.

c. $H \in \sigma(A'(n))$ and $H \notin \sigma(A'(n-1))$ for some $n \in \mathbf{Z}$. Put $\alpha(H) = n$.

Let $B = GG(\alpha)$ and notice that $B'(n) = \bigoplus_{H \in \sigma(A'(n))} H$ which implies $\dim(B') = \dim(A')$ by 6.9. □

The above results mean that every graded group has the same dimension as a graded group whose terms are direct sums of Bockstein groups. Therefore, $\mathcal{DGG}$ forms a set. In the future we will switch to graded groups with terms being direct sums of Bockstein groups.

7. Hurewicz theorems in extension theory

This section deals with analogs of Hurewicz Theorem in extension theory. The most important case is that of simply connected CW complexes. However, our proofs work for a wider class of spaces, namely nilpotent CW complexes, so that is the setup we chose for this section.

Let us start with a result that is closest in spirit to the classical Hurewicz Theorem.

Lemma 7.1. *Let K be a nilpotent pointed CW complex such that $\pi_*(K)$ is Abelian. If G is a Bockstein group and $dim_G(H_*(K)) \geq k$, then $(\pi_*(K) \wedge G)(k)$ is isomorphic to $(H_*(K) \wedge G)(k)$.*

Proof. If $k \leq 1$, then 7.1 is obvious in view of $\pi_1(K) = H_1(K)$. Therefore only $k \geq 2$ is of interest.

Case 1: Consider $G = \mathbf{Z}_{(l)}$, where l is a set of primes. Recall that $\mathbf{Z}_{(l)}$ is the subring of rationals $\mathbf{Q}$ consisting of ratios m/n, where n is not divisible by all $\mathbf{p} \in l$. In particular, $\mathbf{Z}_{(\emptyset)} = \mathbf{Q}$. Use a localizing map $f : K \to K_G$ of pointed CW complexes such that $\pi_*(f)$ is $\pi_*(K) \to \pi_*(K) \otimes G$ and $H_*(f)$ is $H_*(K) \to H_*(K) \otimes G$ (see [2] or [14], also [21] for K simply connected). Now 7.1 follows from the classical Hurewicz Theorem.

Case 2: $G = \mathbf{Z}/\mathbf{p}^\infty$ for some prime $\mathbf{p}$. The exact sequence $0 \to \mathbf{Z} \to \mathbf{Z}[1/\mathbf{p}] \to G \to 0$ yields $0 \to H * G \to H \to H \otimes \mathbf{Z}[1/\mathbf{p}] \to H \otimes G \to 0$ for any Abelian group H. Consider the localizing map $f : K \to L = K_{1/\mathbf{p}}$ such that $\pi_*(f)$ is $\pi_*(K) \to \pi_*(K) \otimes \mathbf{Z}[1/\mathbf{p}]$ and $H_*(f)$ is $H_*(K) \to H_*(K) \otimes \mathbf{Z}[1/\mathbf{p}]$. We may assume that K is a subcomplex of L and f is the inclusion map. The exact sequence

$$\pi_m(K) \to \pi_m(L) \to \pi_m(L, K) \to \pi_{m-1}(K) \to \pi_{m-1}(L)$$

implies existence of an exact sequence

$$0 \to \pi_m(K) \otimes \mathbf{Z}/\mathbf{p}^\infty \to \pi_m(L, K) \to \pi_{m-1}(K) * \mathbf{Z}/\mathbf{p}^\infty \to 0$$

as the cokernel of $\pi_m(K) \to \pi_m(L)$ is $\pi_m(K) \otimes \mathbf{Z}/\mathbf{p}^\infty$ and the kernel of $\pi_{m-1}(K) \to \pi_{m-1}(L)$ is $\pi_{m-1}(K) * \mathbf{Z}/\mathbf{p}^\infty$. Since $\pi_m(K) \otimes \mathbf{Z}/\mathbf{p}^\infty$ is a divisible group, it is a direct summand of $\pi_m(L, K)$ and the above sequence splits. In particular, $\pi_m(L, K) \equiv (\pi_*(K) \wedge \mathbf{Z}/\mathbf{p}^\infty)(m)$ for all m. Similarly, $H_m(L, K) \equiv (H_*(K) \wedge \mathbf{Z}/\mathbf{p}^\infty)(m)$ for all m. Therefore $\pi_m(L, K) = 0$ for

$m \le k-1$ and the Hurewicz homomorphism $\pi_m(L,K) \to H_m(L,K)$ is an isomorphism for $m = k$ and an epimorphism for $m = k+1$.

Case 3: $G = \mathbf{Z/p}$ for some prime $\mathbf{p}$. We will use mod $\mathbf{p}$ homotopy groups $\pi_n(K;\mathbf{Z/p})$ constructed in [17]. In view of Proposition 1.4 on p.3 in [17] one has an exact sequence

$$0 \to \pi_k(K) \otimes \mathbf{Z/p} \to \pi_k(K;\mathbf{Z/p}) \to \pi_{k-1}(K) * \mathbf{Z/p} \to 0$$

and the mod $\mathbf{p}$ Hurewicz Theorem in [17] (see 3.8 on p.12) implies that $\pi_k(K;\mathbf{Z/p}) \equiv H_k(K;\mathbf{Z/p})$. Therefore $\pi_k(K;\mathbf{Z/p})$ is a vector space over $\mathbf{Z/p}$ and the above exact sequence splits showing $(\pi_*(K) \wedge G)(k)$ being isomorphic to $(H_*(K) \wedge G)(k)$. □

The following result is the algebraic version of the Hurewicz theorem in extension theory (for a more geometric version see 7.5). In the case of Abelian CW complexes it has been proved by Shchepin [19] under a different form as a stronger result than 7.5. In our exposition both versions are equivalent in view of 5.8.

Corollary 7.2. *If K is a nilpotent pointed CW complex such that $\pi_*(K)$ is Abelian, then*

$$dim(H_*(K)) = dim(\pi_*(K)).$$

Proof. In view of 6.10 it suffices to show that $\dim_H(H_*(K)) = \dim_H(\pi_*(K))$ for all Bockstein groups H which follows from 7.1. □

Proposition 7.3. *Suppose A and B are two graded groups such that $A \le B$ does not hold. If A is positive, then there is a pointed compactum X of finite dimension such that $dim_A(X) \le 0 < dim_B(X)$.*

Proof. We may assume that $A(n)$ is countable for each n. Since $A \le B$ is false, there is an Abelian group G such that $\dim_G(A) > \dim_G(B) = k$. Let K be the wedge of S^{k+1} and Moore spaces $M(A(n),n)$ for $n \le k$. Notice that $H_n(K;G) = 0$ for $n \le k$. If $B(k) \oplus G \ne 0$, put $P = K(B(k),k+1)$. If $B(k-1) * G \ne 0$, put $P = K(B(k-1),k)$. We are going to use Theorem II of [12]:

Suppose G is an abelian group, $m > 0$ and K is a countable connected CW complex. Then, the following conditions are equivalent:

1. For any CW complex P and any $a \in H_m(P;G) - \{0\}$ there is a compactum X and a map $\pi : X \to P$ such that

$$a \in \mathrm{Im}(\check{H}_m(X;G) \to \check{H}_m(P;G))$$

and K is an absolute extensor of X.

2. $\widetilde{H}_k(K;G) = 0$ for all $k < m$.

Since $H_{k+1}(P;G) \ne 0$, there is a non-trivial map $X \to P$ such that $K \in AE(X)$. In particular, $\dim(X) \le k+1$ which implies $\dim_{A(i)}(X) \le i$ for all $i \ge k+1$. Since $M(A(i),i) \in AE(X)$ for $i \le k$, we get $K(A(i),i) \in AE(X)$ (see [6]). By 5.8 one gets $\dim_A(X) \le 0$. However, since $f : X \to P$ is

non-trivial, either $\dim_{B(k)}(X) > k$ or $\dim_{B(k-1)}(X) > k-1$. Therefore, $\dim_B(X) \leq 0$ does not hold (see 5.8). □

Corollary 7.4. *The following conditions are equivalent for any positive graded groups A and B:*

1. *$dim(A) \leq dim(B)$.*
2. *$dim_A(X) \geq dim_B(X)$ for all pointed compact spaces X.*
3. *$dim_A(X) \geq dim_B(X)$ for all finitely dimensional pointed compacta X.*
4. *$dim_A(X) \leq 0$ implies $dim_B(X) \leq 0$ for every finite-dimensional pointed compactum X.*

Proof. 1) $\Longrightarrow$ 2). Put $C = \mathcal{H}^{-*}(\mathcal{X})$. Since $A \leq B$, $A \wedge C \leq B \wedge C$ and $\dim_B(X) = -\dim_{\mathbf{Z}}(B \wedge C) \leq -\dim_{\mathbf{Z}}(A \wedge C) = \dim_A(X)$.

Both 2) $\Longrightarrow$ 3) and 3) $\Longrightarrow$ 4) are obvious.

4) $\Longrightarrow$ 1) follows from 7.3. □

The following was proved in [6] in the simply connected case and in [3] in the nilpotent case.

Theorem 7.5. *If K is a nilpotent pointed CW complex and X is a compact space of finite dimension, then the following conditions are equivalent:*

1. *$K \in AE(X)$.*
2. *$SP(K) \in AE(X)$.*
3. *$dim_{H_n(K)}(X) \leq n$ for all $n \geq 1$.*
4. *$dim_{\pi_n(K)}(X) \leq n$ for all $n \geq 1$.*

Proof. 1) $\Longrightarrow$ 2) is the same as 1.4.

2) $\Longrightarrow$ 3) is the same as 1.5.

3) $\Longrightarrow$ 4) follows from 7.2, 7.4, and 5.8.

4) $\Longrightarrow$ 1) follows from 1.3. □

8. The dual of graded groups

In this section we will define the dual A^* for each A. A^* is the algebraic manifestation of the duality between compact spaces X and their extension dimension ext–dim(X) (which is represented by a CW complex).

Definition 8.1. Given a graded group A define its **dual** A^* as

$$\bigoplus\{\Sigma^k(H) \mid H \in \mathcal{B}_{\mathcal{G}} \text{ and } \| \geq -\dim_{\mathcal{H}}(\mathcal{A})\}.$$

Proposition 8.2. *A^* is the minimum of $\{B \in \mathcal{DGG} \mid dim_{\mathbf{Z}}(\mathcal{A} \wedge \mathcal{B}) \geq \prime\}$.*

Proof. Suppose $\dim_{\mathbf{Z}}(A \wedge B) \geq 0$ and $B = \bigoplus\limits_{H \in \mathcal{B}_{\mathcal{G}}} \Sigma^{\alpha(H)}(H)$. Since $\dim(B) \leq \dim(\Sigma^{\alpha(H)}(H))$ for each $H \in \mathcal{B}_{\mathcal{G}}$, $\dim_{\mathbf{Z}}(A \wedge \Sigma^{\alpha(H)}(H)) \geq 0$ and $\alpha(H) + \dim_{\mathbf{Z}}(A \wedge H) = \alpha(H) + \dim_H(A) \geq 0$. Thus, $\alpha(H) \geq -\dim_H(A)$ and $\dim(A^*) \leq \dim(\Sigma^{\alpha(H)}(H))$ for each $H \in \mathcal{B}_{\mathcal{G}}$ which means $\dim(A^*) \leq \dim(\bigoplus\limits_{H \in \mathcal{B}_{\mathcal{G}}} \Sigma^{\alpha(H)}(H)) = \dim(B)$. □

Corollary 8.3. *If $dim(A) \leq dim(B)$, then $dim(A^*) \geq dim(B^*)$.*

Proof. $0 \le \dim_{\mathbf{Z}}(A \wedge A^*) \le \dim_{\mathbf{Z}}(B \wedge A^*)$ implies $\dim(A^*) \ge \dim(B^*)$ by the previous result. $\square$

Theorem 8.4. *$\dim((A^*)^*) = \dim(A)$.*

Proof. $0 \le \dim_{\mathbf{Z}}(A \wedge A^*)$ implies $\dim((A^*)^*) \le \dim(A)$ by 8.2.

We need to show $\dim(A) \le \dim((A^*)^*)$. Let $B = (A^*)^*$. It suffices to show that $\dim_H(A) \ge k$ implies $\dim_H((A^*)^*) \ge k$ for all $H \in \mathcal{B}_{\mathcal{G}}$. Since $\dim_H(A) \ge k$, H is a direct summand of $A^*(-k)$. If F is a direct summand of $B(s)$ for $s \le k-1$, then $F \otimes A^*(-k) = 0$ and $F * A^*(-k) = 0$ if $s < k-1$. Therefore, $H \otimes B(s) = 0$ for $s \le k-1$, and $H * B(s) = 0$ if $s < k-1$. This proves $\dim_H(B) \ge k$. $\square$

Given a graded group A we are presented with two problems:

a. to represent A as $GG(\alpha)$ for some optimal $\alpha : \mathcal{B}_{\mathcal{G}} \to \mathbf{Z} \cup \{\pm\infty\}$,

b. to compute dimensions $\dim_H(A)$ of A for all $H \in \mathcal{B}_{\mathcal{G}}$.

It turns out these two problems are intertwined. The natural choice for $\alpha(H)$ in a) would be to seek the smallest integer n so that $\dim(A) \le \dim(\Sigma^n(H))$. This leads to the following concepts.

Definition 8.5. Given a graded group A the function $d_A : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ is defined by $d_A(X) = \dim_{\mathbf{Z}}(A \wedge X)$. The function $e_A : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ is defined as the infimum of all m so that $\dim(A) \le \dim(\Sigma^m(X))$.

Theorem 8.6. *$e_A(X) = -\dim_{\mathbf{Z}}(A^* \wedge X) = -d_{A^*}(X)$ for all graded groups A and X.*

Proof. Suppose $e_A(X) \le m$, i.e. $\dim(A) \le \dim(\Sigma^m(X))$. This implies $0 \le \dim_{\mathbf{Z}}(A \wedge A^*) \le \dim_{\mathbf{Z}}(\Sigma^m(X)) \wedge A^*) = m + \dim_{\mathbf{Z}}(X \wedge A^*)$, i.e. $-\dim_{\mathbf{Z}}(X \wedge A^*) \le m$. Thus, $-\dim_{\mathbf{Z}}(X \wedge A^*) \le e_A(X)$.

Suppose $-\dim_{\mathbf{Z}}(X \wedge A^*) \le k$. This implies $0 \le k + \dim_{\mathbf{Z}}(X \wedge A^*) = \dim_{\mathbf{Z}}(\Sigma^k(X) \wedge A^*)$. Therefore $\dim((A^*)^*) \le \dim(\Sigma^k(X))$ and $e_A(X) \le k$ as $\dim((A^*)^*) = \dim(A)$. This means $e_A(X) \le -\dim_{\mathbf{Z}}(X \wedge A^*)$. $\square$

Definition 8.7. A function $\alpha : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ is called a **dimension-like function** if the following conditions are satisfied:

1. $\alpha(\Sigma^k(A)) = k + \alpha(A)$.
2. $\dim(A) \le \dim(B)$ implies $\alpha(A) \le \alpha(B)$.

3 $\alpha(\bigoplus\limits_{t \in T} A_t) = \inf\limits_{t \in T} \alpha(A_t)$.

α is called an **extension function** if $-\alpha$ is a dimension-like function.

Corollary 8.8. *d_A is a dimension-like function and e_A is an extension function for every graded group A.*

Proof. d_A is clearly a dimension-like function for all A which implies that e_A is an extension function for all A. $\square$

Let us show that extension/dimension-like functions are completely determined by their values on Bockstein groups.

Proposition 8.9. *Given two extension functions $e_1, e_2 : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ the following conditions are equivalent:*
1. *$e_1 \le e_2$.*
2. *$e_1(H) \le e_2(H)$ for each $H \in \mathcal{B}_\mathcal{G}$.*

Proof. Only 2) $\implies$ 1) is of interest. Let $d_1 = -e_1$ and $d_2 = -e_2$ be the corresponding dimension-like functions. We know that $d_2(H) \ge d_1(H)$ for each $H \in \mathcal{B}_\mathcal{G}$ and we need to show $d_2(A) \ge d_1(A)$ for each graded group A. First assume $A = \Sigma^k(H)$ for some $H \in \mathcal{B}_\mathcal{G}$. By 1) of 8.7, $d_2(A) = k + d_2(H) \ge k + d_1(H) = d_1(A)$. By 6.13, any A is equivalent to $\bigoplus\limits_{t\in T} A_t$ so that $d_2(A_t) \ge d_1(A_t)$ for each $t \in T$. Now, 4.26.3 says that $d_2(A) = \inf\limits_{t\in T} d_2(A_t) \ge \inf\limits_{t\in T} d_1(A_t) = d_1(A)$. □

A natural question arises which functions $\alpha : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ give rise to extension/dimension-like functions.

Definition 8.10. $\alpha : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ is a Bockstein function if the following conditions are satisfied:
1. $\alpha(\mathbf{Z}/\mathbf{p}^\infty) \le \alpha(\mathbf{Z}/\mathbf{p})$,
2. $\alpha(\mathbf{Z}/\mathbf{p}) \le \alpha(\mathbf{Z}/\mathbf{p}^\infty) + 1$,
3. $\alpha(\mathbf{Q}) \le \alpha(\mathbf{Z}_{(\mathbf{p})})$,
4. $\alpha(\mathbf{Z}/\mathbf{p}) \le \alpha(\mathbf{Z}_{(\mathbf{p})})$,
5. $\alpha(\mathbf{Z}/\mathbf{p}^\infty) \le \max(\alpha(\mathbf{Q}), \alpha(\mathbf{Z}_{(\mathbf{p})}) - 1)$,
6. $\alpha(\mathbf{Z}_{(\mathbf{p})}) \le \max(\alpha(\mathbf{Q}), \alpha(\mathbf{Z}/\mathbf{p}^\infty) + 1)$.

Proposition 8.11. *If $e : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ is an extension function, then its restriction $e|\mathcal{B}_\mathcal{G} : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ is a Bockstein function.*

Proof. We need to check conditions 1)-6) of 8.10. 1), 3), and 4) follow from 6.11. Notice that $\mathbf{Z}/\mathbf{p}$ is a subgroup of $\mathbf{Z}/\mathbf{p}^\infty$ and one has an exact sequence $0 \to \mathbf{Z}/\mathbf{p} \to \mathbf{Z}/\mathbf{p}^\infty \to \mathbf{Z}/\mathbf{p}^\infty \to 0$. Thus, 2) follows from 6.4. Notice that $\mathbf{Z}_{(\mathbf{p})}$ is a subgroup of $\mathbf{Q}$ and one has an exact sequence $0 \to \mathbf{Z}_{(\mathbf{p})} \to \mathbf{Q} \to \mathbf{Z}/\mathbf{p}^\infty \to 0$. Thus, 5) and 6) follow from 6.4. □

Proposition 8.12. *The following conditions are equivalent for all graded groups A and B:*
1. *$dim(A) \le dim(B)$.*
2. *$d_A \le d_B$.*
3. *$e_A \le e_B$.*

Proof. 1) $\implies$ 2) follows from 8.8. 1) is a special case of 2) as $d_A(G) = \dim_G(A)$. Since $e_A = -d_{A^*}$, 3) is equivalent to $\dim(B^*) \le \dim(A^*)$ which is equivalent to 1) by 8.3 and 8.4. □

9. Algebra of Bockstein functions

Our goal is to show that $\mathcal{DGG}$ is isomorphic to an algebra composed of Bockstein functions (an equivalent approch would be to use dimension-like functions; Bockstein functions are traditionally used in cohomological

dimension theory). To do that we need to show that for every Bockstein function β there is a graded group whose extension function equals β.

Definition 9.1. Given a graded Abelian group A its **Bockstein graded group** $BGG(A)$ is defined as $GG(e_A)$.

Proposition 9.2. *If $\beta : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ is a Bockstein function and $A = GG(\beta)$, then the following conditions are satisfied:*
1. $e_A = \beta$.
2. $d_A(H) = e_A(H)$ for $H = \mathbf{Q}, \mathbf{Z}/\mathbf{p}$.
3. If $\beta(\mathbf{Z}_{(p)}) = \beta(\mathbf{Z}/\mathbf{p}^\infty)$, then $d_A(H) = e_A(H)$ for $H = \mathbf{Z}_{(p)}, \mathbf{Z}/\mathbf{p}^\infty$.
4. If $\beta(\mathbf{Z}_{(p)}) > \beta(\mathbf{Z}/\mathbf{p}^\infty)$, then $d_A(\mathbf{Z}_{(p)}) = \min(\beta(\mathbf{Q}), \beta(\mathbf{Z}/\mathbf{p}^\infty))$ and $d_A(\mathbf{Z}/\mathbf{p}^\infty) = \beta(\mathbf{Z}/\mathbf{p}^\infty) + 1$.

Proof. Clearly, $e_A \le \beta$. Suppose $H \in \mathcal{B}_\mathcal{G}$ and $\beta(H) = m$.

If $H = \mathbf{Q}, \mathbf{Z}/\mathbf{p}$, then $H \otimes F = 0$ for every $F \in \mathcal{B}_\mathcal{G}$ such that $\beta(F) < \beta(H)$ and $0 = H * F$ for every $F \in \mathcal{B}_\mathcal{G}$ such that $\beta(F) < \beta(H) - 1$. Since $H \otimes H \neq 0$, $d_A(H) = e_A(H) = m$ in this case.

Assume $\beta(\mathbf{Z}_{(p)}) = \beta(\mathbf{Z}/\mathbf{p}^\infty) = k$ (which implies $\beta(\mathbf{Q}) = k$) and let $G = \mathbf{Z}_{(p)}$ or $G = \mathbf{Z}/\mathbf{p}^\infty$. Notice that $G \otimes F = 0 = G * F = 0$ for every $F \in \mathcal{B}_\mathcal{G}$ such that $\beta(F) < k$ which implies $d_A(H) = k$ if $H = \mathbf{Z}_{(p)}$ or $H = \mathbf{Z}/\mathbf{p}^\infty$. Also, assuming $G = \mathbf{Z}_{(p)}$, $\dim_G(A) = k > k - 1 = \dim_G(\Sigma^{k-1}(H))$ if $H = \mathbf{Z}_{(p)}$ or $H = \mathbf{Z}/\mathbf{p}^\infty$. Thus, $e_A(H) = \beta(H)$ if $H = \mathbf{Z}_{(p)}$ or $H = \mathbf{Z}/\mathbf{p}^\infty$.

Assume $\beta(\mathbf{Z}_{(p)}) = \beta(\mathbf{Z}/\mathbf{p}^\infty) + 1 = k$ which implies $\beta(\mathbf{Q}) \le k$. Let $F = \mathbf{Z}_{(p)}$ and $G = \mathbf{Z}/\mathbf{p}^\infty$. Notice that $\dim_G(A) = k$ and $\dim_F(A) = \min(\beta(\mathbf{Q}), \beta(\mathbf{Z}/\mathbf{p}^\infty))$. Since $\dim_G(\Sigma^{k-1}(\mathbf{Z}_{(p)})) = k-1$ and $\dim_G(\Sigma^{k-2}(\mathbf{Z}/\mathbf{p}^\infty)) = k - 1$, we get $e_A(\mathbf{Z}_{(p)}) = \beta(\mathbf{Z}_{(p)})$ and $e_A(\mathbf{Z}/\mathbf{p}^\infty) = \beta(\mathbf{Z}/\mathbf{p}^\infty)$.

Assume $k = \beta(\mathbf{Z}_{(p)}) > \beta(\mathbf{Z}/\mathbf{p}^\infty) + 1 = m$ which implies $\beta(\mathbf{Q}) = k$. Let $F = \mathbf{Z}_{(p)}$ and $G = \mathbf{Z}/\mathbf{p}^\infty$. Notice that $\dim_G(A) = m$, $\dim_F(A) = m-1$, and $\dim_\mathbf{Q}(A) = k$. Since $\dim_\mathbf{Q}(\Sigma^{k-1}(\mathbf{Z}_{(p)})) = k - 1$ and $\dim_G(\Sigma^{m-2}(\mathbf{Z}/\mathbf{p}^\infty)) = m - 1$, we get $e_A(\mathbf{Z}_{(p)}) = \beta(\mathbf{Z}_{(p)})$ and $e_A(\mathbf{Z}/\mathbf{p}^\infty) = \beta(\mathbf{Z}/\mathbf{p}^\infty)$. □

Corollary 9.3. *$dim(A) = dim(BGG(A))$ for any graded Abelian group A.*

Proof. Let $B = BGG(A)$. By 9.2, $e_A = e_B$ which implies $\dim(A) = \dim(B)$ by 8.12. □

Corollary 9.4. *If A is graded Abelian group and* **p** *is a prime number, then the following conditions are equivalent:*
1. $e_A(\mathbf{Z}/\mathbf{p}^\infty) = e_A(\mathbf{Z}_{(p)})$.
2. $d_A(\mathbf{Z}/\mathbf{p}^\infty) = d_A(\mathbf{Z}_{(p)})$.
3. $e_A(\mathbf{Z}/\mathbf{p}^\infty) = d_A(\mathbf{Z}/\mathbf{p}^\infty)$.
4. $e_A(\mathbf{Z}_{(p)}) = d_A(\mathbf{Z}_{(p)})$.

Proof. Let $B = BGG(A)$. By 9.3, $\dim(A) = \dim(B)$ which implies 9.4 in view of 9.2. □

Definition 9.5. A Bockstein/dimension-like function β is **p**-regular (respectively, **p**-singular) if $\beta(\mathbf{Z}/\mathbf{p}^\infty) = \beta(\mathbf{Z}_{(p)})$ (respectively, $\beta(\mathbf{Z}/\mathbf{p}^\infty) \neq \beta(\mathbf{Z}_{(p)})$).

A graded group A is $\mathbf{p}$-regular (respectively, $\mathbf{p}$-singular) if e_A is $\mathbf{p}$-regular (respectively, $\mathbf{p}$-singular).

Corollary 9.6. *If A is a graded group, then the following conditions are satisfied:*

1. $d_A(H) = e_A(H)$ for $H = \mathbf{Q}, \mathbf{Z/p}$.

2. If A is $\mathbf{p}$-regular, then $d_A(H) = e_A(H)$ for $H = \mathbf{Z}_{(p)}, \mathbf{Z/p^\infty}$.

3. If A is $\mathbf{p}$-singular, then $d_A(\mathbf{Z}_{(p)}) = \min(e_A(\mathbf{Q}), e_A(\mathbf{Z/p^\infty}))$, $d_A(\mathbf{Z/p^\infty}) = e_A(\mathbf{Z/p^\infty}) + 1$, and $e_A(\mathbf{Z}_{(p)}) = \max(d_A(\mathbf{Q}), d_A(\mathbf{Z/p^\infty}))$.

Proof. Let $B = BGG(A)$. By 9.3, $\dim(A) = \dim(B)$ which implies 9.6 in view of 9.2. □

To be able to add Bockstein functions in a meaningful way, we assume the convention $+\infty + (-\infty) = +\infty$ as $-\infty$ is understood to be the infimum of all integers (see the proof below).

Corollary 9.7. *If A, B are graded groups and $C = A \wedge B$, then the following conditions are satisfied:*

1. $d_C(H) = e_C(H) = e_A(H) + e_B(H)$ for $H = \mathbf{Q}, \mathbf{Z/p}$.

2. If A or B is $\mathbf{p}$-regular, then $d_C(H) = d_A(H) + d_B(H)$ and $e_C(H) = e_A(H) + e_B(H)$ for $H = \mathbf{Z}_{(p)}, \mathbf{Z/p^\infty}$.

3. If both A and B are $\mathbf{p}$-singular, then

$$e_C(\mathbf{Z/p^\infty}) = \min(e_A(\mathbf{Z/p}) + e_B(\mathbf{Z/p}), e_A(\mathbf{Z/p^\infty}) + e_B(\mathbf{Z/p^\infty}) + 1),$$
$$d_C(\mathbf{Z/p^\infty}) = \min(d_A(\mathbf{Z/p}) + d_B(\mathbf{Z/p}) + 1, d_A(\mathbf{Z/p^\infty}) + d_B(\mathbf{Z/p^\infty})),$$
$$e_C(\mathbf{Z_{(p)}}) = \max(e_C(\mathbf{Q}), e_C(\mathbf{Z/p^\infty}) + 1),$$
$$d_C(\mathbf{Z_{(p)}}) = \min(d_C(\mathbf{Q}), d_C(\mathbf{Z/p^\infty}) - 1).$$

Proof. By 9.3, we may assume $A = GG(\alpha)$ and $B = GG(\beta)$, where $\alpha = e_A$ and $\beta = e_B$. We will concentrate on computing d_C as the equations for e_C follow from 9.6. Notice that $C = \bigoplus\limits_{F,G\in\mathcal{B}_\mathcal{G}} \Sigma^{\alpha(F)+\beta(G)}(F \wedge G)$. Therefore, $d_C(H) = \inf\{\alpha(F) + \beta(G) + \dim_H(F \wedge G) \mid F, G \in \mathcal{B}_\mathcal{G}\}$.

If $H = \mathbf{Q}, \mathbf{Z/p}$, then the infimum of $\alpha(F)+\beta(G)+\dim_H(F\wedge G)$ is achieved for $F = G = H$ (see 8.10) and is equal to $\alpha(H) + \beta(H) = d_A(H) + d_B(H)$.

Assume A is $\mathbf{p}$-regular. If $H = \mathbf{Z}_{(p)}$, then the infimum of $\alpha(F) + \beta(G) + \dim_H(F \wedge G)$ is achieved either for $F = G = \mathbf{Q}$ (and equals $d_A(\mathbf{Q}) + d_B(\mathbf{Q}) = d_A(\mathbf{Z}_{(p)}) + d_B(\mathbf{Q})$) or is achieved for $F = \mathbf{Z}_{(p)}$, $G = \mathbf{Z/p^\infty}$ (see 8.10) and is equal to $\alpha(F) + \beta(G) = d_A(\mathbf{Z}_{(p)}) + e_B(\mathbf{Z/p^\infty})$. Thus, $d_H(C) = d_A(\mathbf{Z}_{(p)}) + \min(d_B(\mathbf{Q}), e_B(\mathbf{Z/p^\infty})) = d_A(\mathbf{Z}_{(p)}) + d_B(\mathbf{Z}_{(p)})$. If $H = \mathbf{Z/p^\infty}$, then the infimum of $\alpha(F) + \beta(G) + \dim_H(F \wedge G)$ is achieved either for $F = \mathbf{Z}_{(p)}$, $G = \mathbf{Z/p^\infty}$ (see 8.10) and is equal to $\alpha(F) + \beta(G) + 1$, or for $F = \mathbf{Z}_{(p)}$, $G = \mathbf{Z}_{(p)}$ (see 8.10) and is equal to $\alpha(F) + \beta(G)$. Thus, $d_H(C) = d_A(\mathbf{Z}_{(p)}) + \min(e_B(\mathbf{Z}_{(p)}), e_B(\mathbf{Z/p^\infty}) + 1) = d_A(\mathbf{Z/p^\infty}) + d_B(\mathbf{Z/p^\infty})$.

Assume both A and B are $\mathbf{p}$-singular. If $H = \mathbf{Z/p^\infty}$, then the infimum of $\alpha(F) + \beta(G) + \dim_H(F \wedge G)$ is achieved either for $F = \mathbf{Z/p}$, $G = \mathbf{Z/p}$

(see 8.10) and is equal to $\alpha(F)+\beta(G)+1 = d_A(\mathbf{Z/p})+d_B(\mathbf{Z/p})+1$, or for $F = \mathbf{Z/p^\infty}$, $G = \mathbf{Z/p^\infty}$ (see 8.10) and is equal to $\alpha(F)+\beta(G)+2 = d_A(H)+d_B(H)$. Thus, $d_C(\mathbf{Z/p^\infty}) = \min(d_A(\mathbf{Z/p}) + d_B(\mathbf{Z/p}) + 1, d_A(\mathbf{Z/p^\infty}) + d_B(\mathbf{Z/p^\infty}))$. If $H = \mathbf{Z}_{(p)}$, then the infimum of $\alpha(F) + \beta(G) + \dim_H(F \wedge G)$ is achieved either for $F = G = \mathbf{Q}$ and equals $d_A(\mathbf{Q}) + d_B(\mathbf{Q}) = d_C(\mathbf{Q})$, or is achieved for $F = \mathbf{Z/p^\infty}$, $G = \mathbf{Z/p^\infty}$ and is equal to $\alpha(F) + \beta(G) + 1$, or is achieved either for $F = \mathbf{Z/p}$, $G = \mathbf{Z/p}$ and is equal to $\alpha(F) + \beta(G) = d_A(\mathbf{Z/p}) + d_B(\mathbf{Z/p})$. From what we know about $d_C(\mathbf{Z/p^\infty})$ we conclude $d_C(\mathbf{Z_{(p)}}) = \min(d_C(\mathbf{Q}), d_C(\mathbf{Z/p^\infty}) - 1)$. □

Definition 9.8. Given two Bockstein functions $\alpha : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ and $\beta : \mathcal{B}_\mathcal{G} \to \mathbf{Z}$ define their **smash product** $\alpha \wedge \beta$ as the extension function of $GG(\alpha) \wedge GG(\beta)$.

The next result is a corollary to 9.7 and it shows that our definition of the smash product of Bockstein functions coincides with the one given in [9].

Theorem 9.9. *Given two Bockstein functions $\alpha : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ and $\beta : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ their smash product $\alpha \wedge \beta$ is given by the following formulae:*

1. Let $\gamma = \alpha + \beta$,

2. If $H \neq \mathbf{Z/p^\infty}$ and $H \neq \mathbf{Z_{(p)}}$ for all $\mathbf{p}$, then

$$(\alpha \wedge \beta)(H) = \gamma(H),$$

3. If one of α, β is $\mathbf{p}$-regular for some $\mathbf{p}$, then

$$(\alpha \wedge \beta)(H) = \gamma(H)$$

for $H = \mathbf{Z/p^\infty}, \mathbf{Z_{(p)}}$.

4. If both α, β are $\mathbf{p}$-singular for some $\mathbf{p}$, then

$$(\alpha \wedge \beta)(\mathbf{Z/p^\infty}) = \min(\gamma(\mathbf{Z/p}), \gamma(\mathbf{Z/p^\infty}) + 1),$$
$$(\alpha \wedge \beta)(\mathbf{Z_{(p)}}) = \max(\gamma(\mathbf{Q}), (\alpha \wedge \beta)(\mathbf{Z/p^\infty}) + 1).$$

Definition 9.10. Given a Bockstein function $\beta : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ define its **dual** β^* as the extension function of $(GG(\beta))^*$.

Theorem 9.11. *Given a Bockstein function $\beta : \mathcal{B}_\mathcal{G} \to \mathbf{Z} \cup \{\pm\infty\}$ its dual β^* is given by the following formulae:*

1. $\beta^*(H) = -\beta(H)$ for $H = \mathbf{Q}, \mathbf{Z/p}$.

2. If β is $\mathbf{p}$-regular for some $\mathbf{p}$, then $\beta^*(H) = -\beta(H)$ for $H = \mathbf{Z/p^\infty}, \mathbf{Z}_{(p)}$.

3. If β is $\mathbf{p}$-singular for some $\mathbf{p}$, then $\beta^*(\mathbf{Z/p^\infty}) = -\beta(\mathbf{Z/p^\infty}) - 1$ and $\beta^*(\mathbf{Z}_{(p)}) = \max(-\beta(\mathbf{Q}), -\beta(\mathbf{Z/p^\infty}))$.

Proof. Let $A = GG(\beta)$ and $B = A^*$. By 8.6, $d_B = -e_A = -\beta$. Using 9.6 one arrives at 1)-3). □

Theorem 9.12. *Let $\mathcal{BF}$ be the family of all Bockstein functions. If β is a Bockstein function, then*

$$\beta^* = \inf\{\alpha \in \mathcal{BF} \mid \alpha \wedge \beta \geq \prime\}.$$

Proof. Notice that $\beta^* \wedge \beta \geq 0$. Suppose $\alpha \wedge \beta \geq 0$ and let $A = GG(\alpha)$, $B = GG(\beta^*)$. Since $\dim_{\mathbf{Z}}(A \wedge B^*) \geq 0$, $\dim(B) \leq \dim(A)$ (see 8.2) and $\beta^* = e_B \leq e_A = \alpha$ (see 8.12 and 9.2). $\square$

Definition 9.13. Given two Bockstein functions α and β, one defines their **sum–product** $\alpha[+]\beta$ as follows:

1. $(\alpha[+]\beta)(\mathbf{Z/p}) = \alpha(\mathbf{Z/p}) + \beta(\mathbf{Z/p})$,
2. $(\alpha[+]\beta)(\mathbf{Q}) = \alpha(\mathbf{Q}) + \beta(\mathbf{Q})$,
3. $(\alpha[+]\beta)(\mathbf{Z/p^\infty}) = \max\{\alpha(\mathbf{Z/p^\infty}) + \beta(\mathbf{Z/p^\infty}), \alpha(\mathbf{Z/p}) + \beta(\mathbf{Z/p}) - 1\}$,
4. $(\alpha[+]\beta)(\mathbf{Z_{(p)}}) = \alpha(\mathbf{Z_{(p)}}) + \beta(\mathbf{Z_{(p)}})$ if $\alpha(\mathbf{Z_{(p)}}) = \alpha(\mathbf{Z/p^\infty})$ or $\beta(\mathbf{Z_{(p)}}) = \beta(\mathbf{Z/p^\infty})$,

   $(\alpha[+]\beta)(\mathbf{Z_{(p)}}) = \max\{(\alpha[+]\beta)(\mathbf{Z/p^\infty}) + 1, \alpha(\mathbf{Q}) + \beta(\mathbf{Q})\}$ if $\alpha(\mathbf{Z_{(p)}}) > \alpha(\mathbf{Z/p^\infty})$ and $\beta(\mathbf{Z_{(p)}}) > \beta(\mathbf{Z/p^\infty})$.

Remark 9.14. $\alpha[+]\beta$ was introduced in [16] under the name of the product of two Bockstein functions and is denoted there by $\alpha \times \beta$. A.Dranishnikov [7] realized that $\alpha[+]\beta$ is much closer related to the sum $\alpha + \beta$ and that there is another operation, denoted by $\alpha[\times]\beta$, which should play the role of a product.

Theorem 9.15. $\alpha[+]\beta = (\alpha^* \wedge \beta^*)^*$ *for every two Bockstein functions* α *and* β.

Proof. Let $\gamma = \alpha[+]\beta$ and $\nu = (\alpha^* \wedge \beta^*)^*$. Applying 9.9 and 9.11 we get $\gamma(H) = \nu(H)$ if $H = \mathbf{Z/p}, \mathbf{Q}$.

If both α and β are **p**-regular, then γ and ν are **p**-regular and $\gamma(H) = \nu(H) = \alpha(H) + \beta(H)$ for $H = \mathbf{Z}_{(p)}, \mathbf{Z/p^\infty}$.

If both α and β are **p**-singular, then both α^* and β^* are **p**-singular. By 9.11, $\alpha^*(\mathbf{Z/p^\infty}) = -\alpha(\mathbf{Z/p^\infty}) - 1$ and $\beta^*(\mathbf{Z/p^\infty}) = -\beta(\mathbf{Z/p^\infty}) - 1$. By 9.9, $(\alpha^* \wedge \beta^*)(\mathbf{Z/p^\infty}) = \min(-\alpha(\mathbf{Z/p}) - \beta(\mathbf{Z/p}), -\alpha(\mathbf{Z/p^\infty}) - \beta(\mathbf{Z/p^\infty}) - 1)$. Applying 9.9 again, we get $\nu(\mathbf{Z/p^\infty}) = \max(\alpha(\mathbf{Z/p}) + \beta(\mathbf{Z/p}), \alpha(\mathbf{Z/p^\infty}) + \beta(\mathbf{Z/p^\infty}) + 1) - 1 = \gamma(\mathbf{Z/p^\infty})$. Now, both γ and ν are **p**-singular and agree on $\mathbf{Q}$ and $\mathbf{Z/p^\infty}$ which implies that they agree on $\mathbf{Z}_{(p)}$.

Assume α is **p**-regular and β is **p**-singular. Now, α^* is **p**-regular and β^* is **p**-singular. By 9.11, $\beta^*(\mathbf{Z/p^\infty}) = -\beta(\mathbf{Z/p^\infty}) - 1$, $\beta^*(\mathbf{Z}_{(p)}) = \max(-\beta(\mathbf{Q}), -\beta(\mathbf{Z/p^\infty}))$, and $\alpha^*(\mathbf{Z/p^\infty}) = -\alpha(\mathbf{Z/p^\infty}) = \alpha^*(\mathbf{Z}_{(p)})$. Applying 9.9 we get $(\alpha^* \wedge \beta^*)(H) = \alpha^*(H) + \beta^*(H)$ for $H = \mathbf{Z}_{(p)}, \mathbf{Z/p^\infty}$. By 9.11, $\nu(\mathbf{Z/p^\infty}) = -\alpha^*(\mathbf{Z/p^\infty}) - \beta^*(\mathbf{Z/p^\infty}) - 1 = \alpha(\mathbf{Z/p^\infty}) + \beta(\mathbf{Z/p^\infty}) = \gamma(\mathbf{Z/p^\infty})$. Again, both γ and ν are **p**-singular and agree on $\mathbf{Q}$ and $\mathbf{Z/p^\infty}$ which implies that they agree on $\mathbf{Z}_{(p)}$. $\square$

Theorem 9.16. *For any dimension-like function* $d : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ *there is a graded group* A *such that* $d(X) = dim_{\mathbf{Z}}(A \wedge X)$.

Proof. $e = -d$ is an extension function and there is a graded group B so that $e(H) = e_B(H)$ for each $H \in \mathcal{B}_{\mathcal{G}}$. Let $A = B^*$. Now, $d_A(H) = -e_B(H) = d(H)$ for each $H \in \mathcal{B}_{\mathcal{G}}$ which means $d_A = d$. $\square$

Theorem 9.17. *A function $\alpha : \mathcal{B}_{\mathcal{G}} \to \mathbf{Z} \cup \{\pm\infty\}$ extends to an extension function $e : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ if and only if α is a Bockstein function.*

Proof. By 8.11, a restriction $e \mid \mathcal{B}_{\mathcal{G}}$ of an extension function $e : \mathcal{DGG} \to \mathbf{Z} \cup \{\pm\infty\}$ is a Bockstein function. Suppose $\beta : \mathcal{B}_{\mathcal{G}} \to \mathbf{Z} \cup \{\pm\infty\}$ is a Bockstein function. Let $A = GG(\beta)$ be the associated graded group. 9.2 says that $\beta = e_a \mid \mathcal{B}_{\mathcal{G}}$ and e_A is an extension function by 8.8. $\square$

Theorem 9.18. *Any set $\{A_t\}_{t\in T}$ of $\mathcal{DGG}$ has infimum $\bigoplus\limits_{t\in T} A_t$ and supremum $(\bigoplus\limits_{t\in T} A_t^*)^*$.*

Proof. Clearly, $\dim(\bigoplus\limits_{t\in T} A_t) \le \dim(A_t)$ for each $t \in T$. If $\dim(B) \le \dim(A_t)$ for each $t \in T$, then it is easy to show that $\dim(B) \le \dim(\bigoplus\limits_{t\in T} A_t)$.

Dually, $\dim(\bigoplus\limits_{t\in T} A_t^*) \le \dim(A_t^*)$ for each $t \in T$ which implies $\dim((\bigoplus\limits_{t\in T} A_t^*)^*) \ge \dim(A_t)$ for each $t \in T$. If $\dim(B) \ge \dim(A_t)$ for each $t \in T$, then $\dim(B^*) \le \dim(A_t^*)$ and $\dim(B^*) \le \dim(\bigoplus\limits_{t\in T} A_t^*)$. Therefore, $\dim(B) \ge \dim((\bigoplus\limits_{t\in T} A_t^*)^*)$. $\square$

Corollary 9.19. *$\mathcal{DGG}$ is a lattice. Moreover,* $\min(A, B) = A \oplus B$ *and* $\max(A, B) = (A^* \oplus B^*)^*$.

It would be interesting to characterize self-dual graded groups.

Proposition 9.20. *Suppose A is a graded group. $dim(A) = dim(A^*)$ if and only if $dim(A) = dim(\mathbf{Z})$.*

Proof. Notice that $e_{\mathbf{Z}} = 0$ which implies $e_{\mathbf{Z}^*} = 0$ by 9.11. Thus, $\dim(\mathbf{Z}) = \dim(\mathbf{Z}^*)$. If $\dim(A) = \dim(A^*)$, then setting $\beta = e_A$ one gets $\beta = \beta^*$ which is possible only if $\beta = 0$ (see 9.11), i.e. $\dim(A) = \dim(\mathbf{Z})$ (see 8.12). $\square$

10. Applications

Definition 10.1. Suppose K is a pointed CW complex. Given an Abelian group G define define $e^K(G)$ as $e_A(G)$, where $A = H_*(K)$.

Functions e^K were first introduced in [9] to study extension properties of CW complexes via so-called Dual Bockstein Algebra. The idea was to dualize the approach of [16] where Bockstein functions d_X were used to investigate cohomological dimension of compact spaces X. Thus, one has two families of Bockstein functions $d_X, e^K : \mathcal{B}_{\mathcal{G}} \to \mathbf{Z} \cup \{\pm\infty\}$.

Corollary 10.2. *Suppose X is a pointed compact space and K is a pointed CW complex. The following conditions are equivalent:*

1. *$SP(K) \in AE(X)$.*
2. *$d_X \le e^K$.*

Proof. By 5.9 $SP(K) \in AE(X)$ if and only if $\mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}_*(\mathcal{K})$ is non-negative. Therefore 1) is equivalent to $(\mathcal{H}^{-*}(\mathcal{X}))^* \leq \mathcal{H}_*(\mathcal{K})$ which is equivalent to 2). □

The following result was proved in [9] (see Theorem 5.20) for countable CW complexes.

Corollary 10.3. *If K, L are pointed CW complexes, then $e^{K\wedge L} = e^K \wedge e^L$.*

Proof. $e^K \wedge e^L$ was defined in 9.8 as the extension function of $GG(e^K) \wedge GG(e^L)$. By 9.1 and 9.3, $GG(e^M)$ has the same dimension as $H_*(M)$ for each CW complex M. Thus, $e^K \wedge e^L$ is the extension function of $H_*(K) \wedge H_*(L)$ which is $e^{K\wedge L}$ in view of 5.1. □

Definition 10.4. Given a function $\alpha : \mathcal{B}_{\mathcal{G}} \to [\infty, \infty]$ define the Moore space $M(\alpha)$ as the wedge of those $M(H, \alpha(H))$ for which $\alpha(H)$ is finite.

The following result was proved in [13] using different methods.

Theorem 10.5. *Suppose X is a pointed compact space. There is a minimum $\{SP(K) \mid SP(K) \in AE(X)\}$ called the* **cohomological dimension** *of X. That minimum is represented by a countable CW complex.*

Proof. If X is totally disconnected, then $S^0 \in AE(X)$ and $SP(S^0) \sim S^0$ is the minimum of all CW complexes K so that $K \in AE(X)$. Suppose X is not totally disconnected. Now, $A = (\mathcal{H}^{-*}(\mathcal{X}))^*$ is positive and we may consider $K = M(A')$, where A' is built of Bockstein groups and $\dim(A') = \dim(A)$. Notice that $d_X = e^K$ which implies $SP(K) \in AE(X)$ in view of 10.2. If $SP(L) \in AE(X)$ for some L, then $e_K = d_X \leq e_L$ and $H_*(K) \leq H_*(L)$ which implies $SP(K) \leq SP(L)$ (see 10.2). □

Theorem 10.6 (First Bockstein Theorem). *For every compact space X and every Abelian group $G \neq 0$*

$$dim_G(X) = \sup\{dim_H(X) \mid H \in \sigma(G)\}.$$

Proof. Let $F = \bigoplus\limits_{H\in\sigma(G)} H$. By 6.9, $\dim_G(X) = \dim_F(X)$ and it is clear that $\dim_F(X) = \sup\{\dim_H(X) \mid H \in \sigma(G)\}$. □

Theorem 10.7 (Second Bockstein Theorem). *Suppose X and Y are pointed compact spaces. Then,*

1. $dim_{\mathbf{Z/p}}(X \wedge Y) = dim_{\mathbf{Z/p}}X + dim_{\mathbf{Z/p}}Y$,
2. $dim_{\mathbf{Q}}(X \wedge Y) = dim_{\mathbf{Q}}X + dim_{\mathbf{Q}}Y$,
3. $dim_{\mathbf{Z/p^\infty}}(X\wedge Y) = \max\{dim_{\mathbf{Z/p^\infty}}X + dim_{\mathbf{Z/p^\infty}}Y, dim_{\mathbf{Z/p}}X + dim_{\mathbf{Z/p}}Y - 1\}$,
4. $dim_{\mathbf{Z_{(p)}}}(X \wedge Y) = dim_{\mathbf{Z_{(p)}}}X + dim_{\mathbf{Z_{(p)}}}Y$ *if* $dim_{\mathbf{Z_{(p)}}}X = dim_{\mathbf{Z/p^\infty}}X$ *or* $dim_{\mathbf{Z_{(p)}}}Y = dim_{\mathbf{Z/p^\infty}}Y$,

$dim_{\mathbf{Z_{(p)}}}(X\wedge Y) = \max\{dim_{\mathbf{Z/p^\infty}}(X\times Y)+1, dim_{\mathbf{Q}}X + dim_{\mathbf{Q}}Y\}$ *if* $dim_{\mathbf{Z_{(p)}}}X > dim_{\mathbf{Z/p^\infty}}X$ *and* $dim_{\mathbf{Z_{(p)}}}Y > dim_{\mathbf{Z/p^\infty}}Y$.

Proof. Let $A = \mathcal{H}^{-*}(\mathcal{X})$, $B = \mathcal{H}^{-*}(\mathcal{Y})$, and $C = \mathcal{H}^{-*}(\mathcal{X} \wedge \mathcal{Y})$. Notice that $\dim(C) = \dim(A \wedge B) = \dim((A^*)^* \wedge (B^*)^*)$ (see 5.4) which implies $e_{C^*} = e_{A^*}[+]e_{B^*}$. Since $d_X = -d_A = e_{A^*}$, $d_Y = -d_B = e_{B^*}$, and $d_{X\wedge Y} = -d_C = e_{C^*}$, 10.7 follows from 9.15. $\square$

Theorem 10.8 (Dranishnikov Realization Theorem)**.** *Suppose $n \geq 1$ and $\alpha : \mathcal{B}_{\mathcal{G}} \to [\infty, \backslash]$ is a Bockstein function. There is a compactum $X \subset I^{n+2}$ such that $d_X = \alpha$.*

Proof. Let $A = GG(\alpha)$, $B = \Sigma^{n+1}(A^*)$, and $\beta = e_B$. Notice that $\beta \geq 1$. Indeed, using 9.11 one gets $(\alpha)^* \geq -n$ and it is clear that $\beta = (\alpha)^* + n + 1$ which implies $\alpha \wedge \beta \geq n+1$. Let $K = M(\alpha)$ and $L = M(\beta)$. Notice that the extension function of $K \wedge L$ is $\alpha \wedge \beta \geq n+1$ which means that the extension function of $K * L$ is at least $n+2$, i.e. $K * L \in AE(I^{n+2})$. Split I^{n+2} as $X' \cup Y$ so that X' is σ-compact, $K \in AE(X')$, and $L \in AE(Y)$. Replace X' by the compact space X as follows: if X' is the union of its compact subspaces X_n, $n \geq 1$, then X is the compact wedge of all X_n. In particular, X' and X have the same extension dimension (that means $K \in AE(X)$ is equivalent to $K \in AE(X')$ for all CW complexes K) and $d_X \leq \alpha$. Let $d_X = \gamma$ and choose M with extension function γ. Now, $M * L \in AE(I^{n+2})$, i.e. $\gamma \wedge \beta \geq n+1$ and $\gamma \geq (\beta)^* + n + 1 = \alpha$. This proves $d_X = \alpha$. $\square$

Corollary 10.9. *For every Bockstein function $\alpha : \mathcal{B}_{\mathcal{G}} \to [\infty, \infty]$ there is a pointed compactum X such that $d_X = \alpha$.*

Proof. For each $n \geq 1$ put $\alpha_n = \min(\alpha, n)$ and pick a pointed compactum $X(n)$ so that $d_{X(n)} = \alpha_n$. The compact wedge X of all X_n is the required pointed compactum. $\square$

Theorem 10.10 (Test Spaces Theorem)**.** *For each $n > 1$ and each Abelian group G there is a pointed compactum $T \subset I^{n+2}$ such that $dim_{\mathbf{Z}}(X \wedge T) = dim_G(X) + n$ for every pointed compact space X satisfying $dim_{\mathbf{Z}}(X) < dim_G(X) + n$.*

Proof. Consider the graded group $B = \Sigma^{-n}(G) \oplus \Sigma^{-1}(\mathbf{Z})$. We need to find a pointed compactum T so that $\dim(B) = \dim(\mathcal{H}^{-*}(\mathcal{T}))$. Consider $A = B^*$. Notice that $-n \leq d_B \leq -1$ which implies $1 \leq e_A \leq n$ (see 8.6). In particular, A is positive. Choose $T \subset I^{n+2}$ with $d_T = e_A$ (see 10.8) which implies $\dim(\mathcal{H}^{-*}(\mathcal{T})) = \dim(\mathcal{B})$. If $\dim_G(X) = k$ and $\dim_{\mathbf{Z}}(X) < k+n$, then $C = \mathcal{H}^{-*}(\mathcal{X}; \mathcal{G})$ satisfies $C(-k) \neq 0$ and $C(i) = 0$ for $i < -k$. Look at $D = \mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}^{-*}(\mathcal{T})$ and notice that $D(-k-n) \neq 0$ and $D(i) = 0$ for $i < -k-n$. That means $\dim_{\mathbf{Z}}(X \wedge T) = k + n$ by 5.4. $\square$

Remark 10.11. Notice that 10.10 is slightly more general than 6.1 of [8] in the sense that it deals with integral dimension instead of the covering dimension. Also notice that 6.1 of [8] is a significant improvement of the original Test Spaces Theorem (see Theorem 12 in [16]).

11. Embeddings of algebras in $\mathcal{DGG}$

In this section we will discuss embeddings of geometrically defined algebras in the Dimension Algebra of Graded Groups.

Proposition 11.1. *Let $\mathcal{CW}$ be the subalgebra of the Standard Algebra consisting of all pointed CW complexes. Assigning $dim(H_*(K))$ to K is a homomorphism from $\mathcal{CW}$ to $\mathcal{DGG}$ whose image consists of $dim(A)$ such that A is a non-negative graded group and $A(0)$ is free Abelian.*

Proof. Clearly, $H_*(K \vee L) \equiv H_*(K) \oplus H_*(L)$. Also $H_*(K \wedge L) \equiv H_*(K) \wedge H_*(L)$ by 5.1.

Obviously, $\dim(H_*(K))$ is a non-negative graded group whose 0-th term is free Abelian. Suppose A is a non-negative graded group and $A(0)$ is free Abelian.

Case 1. $A(0) = 0$. In this case one can consider the Moore space $M_n = M(A_n, n)$ (a pointed CW complex with only one non-zero homology and whose n-th homology is A_n). The wedge M of all M_n, $n \geq 1$, is a CW complex such that $H_*(M) = A$.

Case 2. $A(0) \neq 0$. In this case it is obvious that $\dim(A) = \dim(Z)$ and $M = S^0$ satisfies $\dim(H_*(M)) = \dim(A)$. □

Proposition 11.2. *Let $\mathcal{COMP}$ be the subalgebra of the Standard Algebra consisting of all pointed compact spaces. Assigning $dim(\mathcal{H}^{-*}(\mathcal{X}))$ to X is a homomorphism from $\mathcal{COMP}$ to $\mathcal{DGG}$ whose image consists of $dim(0)$ and $dim(A^*)$ such that A is a non-negative graded group and $A(0)$ is free Abelian.*

Proof. Clearly, $\mathcal{H}^{-*}(\mathcal{X} \vee \mathcal{Y}) \equiv \mathcal{H}^{-*}(\mathcal{X}) \oplus \mathcal{H}^{-*}(\mathcal{Y})$. Also $\mathcal{H}^{-*}(\mathcal{X} \wedge \mathcal{Y}) \equiv \mathcal{H}^{-*}(\mathcal{X}) \wedge \mathcal{H}^{-*}(\mathcal{Y})$ by 5.4.

If X is a pointed point, then $\mathcal{H}^{-*}(\mathcal{X}) = \prime$. If X contains at least two points, then put $B = (H^{-*}(X))^*$. Notice that $e_B = d_X$, so it is either positive or identically 0 (see 5.6). Since $C = GG(e_B)$ satisfies $\dim(C) = \dim(B)$ by 9.3, C is positive if $e_B > 0$ or $\dim(C) = \dim(\mathbf{Z})$ if $e_B \equiv 0$. Putting $A = C$ if if $e_B > 0$ and $A = \mathbf{Z}$ if $e_B \equiv 0$ one gets a non-negative graded group A such that $A(0)$ is free and $\dim(A^*) = \dim(\mathcal{H}^{-*}(\mathcal{X}))$.

Conversely, if A is a non-negative graded group such that $A(0)$ is Abelian, then e_A is a Bockstein function which is either positive or identically 0 if $A(0) \neq 0$. Therefore, by 10.9, there is a pointed compactum X such that $d_X = e_A$. That amounts to $\dim(\mathcal{H}^{-*}(\mathcal{X})) = \dim(\mathcal{A}^*)$ by 8.6. □

Let us give sufficient and necessary conditions for two pointed CW complexes in $\mathcal{CW}$ to be mapped to the same element of $\mathcal{DGG}$. The result below was proved in [13] for countable CW complexes using different methods.

Proposition 11.3. *If K and L are pointed CW complexes, then the following conditions are equivalent:*

1. *$dim(H_*(K)) = dim(H_*(L))$.*
2. *$cin(K \wedge M) = cin(L \wedge M)$ for all pointed CW complexes M.*

3. There is $n \geq 2$ such that $\Sigma^n(K) \sim_X \Sigma^n(L)$ for all finite-dimensional compacta X.

4. $SP(K) \sim_X SP(L)$ for all compact spaces X.

Proof. 1) $\Longrightarrow$ 2). Notice that $cin(P \wedge M) = \dim_{\mathbf{Z}}(H_*(P) \wedge H_*(M))$ for all pointed CW complexes M and P. Using 5.1, 1) $\Longrightarrow$ 2) follows.

2) $\Longrightarrow$ 1). Given an abelian group G consider $M = M(G,1)$, the Moore space. Thus, $H_*(P) = \Sigma(G)$ and $cin(P \wedge M) = \dim_{\mathbf{Z}}(H_*(P) \wedge \Sigma(G)) = 1 + \dim_{\mathbf{Z}}(H_*(P) \wedge G) = 1 + \dim_G(H_*(P))$. Thus $1 + \dim_G(H_*(K)) = cin(K \wedge M) = cin(L \wedge M) = 1 + \dim_G(H_*(L))$ which means $\dim_G(H_*(K)) = \dim_G(H_*(L))$ for all G, i.e. $\dim(H_*(K)) = \dim(H_*(L))$.

1) $\Longleftrightarrow$ 3) and 1) $\Longleftrightarrow$ 4) follow from 7.5 and 7.4. $\square$

The following two results are immediate corollaries of 11.3 and 11.1.

Theorem 11.4. *Assigning dim$(H_*(K))$ to the equivalence class of a pointed CW complex K induces an embedding of Shchepin Algebra into $\mathcal{DGG}$.*

Theorem 11.5. *Assigning dim$(H_*(K))$ to the equivalence class of a pointed CW complex K induces an embedding of Dranishnikov-Dydak Algebra into $\mathcal{DGG}$.*

Corollary 11.6. *Dranishnikov-Dydak Algebra and Shchepin Algebra are identical.*

Proof. In view of the above results one has the natural embedding of Dranishnikov-Dydak Algebra to Shchepin Algebra. Notice that 6.13 implies that for any pointed CW complex K there is a pointed countable CW complex K' with $\dim(H_*(K)) = \dim(H_*(K'))$. Therefore, the embedding of algebras is surjective. $\square$

Theorem 11.7. *Assigning dim$(\mathcal{H}^{-*}(\mathcal{X}))$ to X induces an embedding of Kuzminov Algebra into $\mathcal{DGG}$.*

Proof. Suppose X and Y are two pointed compact spaces of finite dimension such that $\dim(X \wedge T) = \dim(Y \wedge T)$ for all pointed compact spaces T of finite dimension. To show $\dim(\mathcal{H}^{-*}(\mathcal{X})) = \dim(\mathcal{H}^{-*}(\mathcal{Y}))$ it suffices to prove $\dim_G(X) = \dim_G(Y)$ for all Abelian groups G. Pick $n > \dim(X), \dim(Y)$ and use 10.10 to produce T such that $\dim(X \wedge T) = \dim_G(X) + n$ and $\dim(Y \wedge T) = \dim_G(Y) + n$. Hence $\dim_G(X) = \dim_G(Y)$.

Conversely, if $\dim(\mathcal{H}^{-*}(\mathcal{X})) = \dim(\mathcal{H}^{-*}(\mathcal{Y}))$, i.e. $\dim_G(X) = \dim_G(Y)$ for all Abelian groups G, then 10.7 implies $\dim_H(X \wedge T) = \dim_H(Y \wedge T)$ for all Bockstein groups H. Applying 10.6 one gets $\dim_{\mathbf{Z}}(X \wedge T) = \dim_{\mathbf{Z}}(Y \wedge T)$ which implies $\dim(X \wedge T) = \dim(Y \wedge T)$ if T is finite-dimensional. $\square$

Proposition 3.7 of [9] gives the following improvement of 3.2. In contrast to 3.2 that result requires algebra which is quite easy in our setting.

Lemma 11.8. *Suppose K_1, K_2, L_1, L_2 are pointed countable CW complexes. If $K_i \sim_X L_i$ for $i = 1,2$ and for all finite-dimensional compacta X, then $K_1 \wedge L_1 \sim_X K_2 \wedge L_2$ for all finite-dimensional compacta X.*

Proof. As in Proposition 3.7 of [9] the only non-geometric case is that of connected CW complexes. In that case $K_1 \wedge L_1 \sim_X K_2 \wedge L_2$ for all finite-dimensional compacta X if $H_*(K_1 \wedge L_1)$ and $H_*(K_2 \wedge L_2)$ represent the same element of $\mathcal{DGG}$. However, that is true as 3.2 and 7.5 imply that their suspensions represent the same element of $\mathcal{DGG}$ (see 7.5). □

Therefore the following definition makes sense.

Definition 11.9. The **Unstable Extension Algebra** is the quotient of the Standard Algebra of pointed countable CW complexes under the equivalence relation $K \sim_X L$ for all finite-dimensional compacta.

Notice that the above algebra is not identical with the Dranishnikov-Dydak Algebra. Indeed, any $M = M(G, 1)$, where $G \neq 0$ is perfect, is not equivalent to I as $M \notin AE(I^2)$ and $I \in AE(I^2)$. On the other hand $\Sigma(M)$ and $\Sigma(I)$ are both contractible.

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